Numerical Methods in the Theory of Topological Solitons

Student Report

submitted by

Roman Sergeevich Kirichenkov

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Supervisor

Dr. Yakov M. Shnir

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Contents

1. Introduction

1.1. General information

Soliton is a solution of the field equations which represents a particle-like configuration of the field with specific properties. Common properties of the configuration are permanence of the form, localization within a region and a possibility to interact with other solitons or fields.

In general construction of solitons comes down to consideration of different Lagrangian densities with or without constraints on field potentials and defining a certain type of field potentials that can create a soliton solution.

1.2. Description of a model

The model of a field that we will consider is a 2+1 dim O(3) sigma model with the Lagrangian density that is determined by expression:

$$\mathcal{L} = \frac{1}{4} \left(\partial_\mu \varphi^a \cdot \partial^\mu \varphi^a\right)$$

and a constraint $\varphi^a \cdot \varphi^a = 1$ on the triplet of the fields $\varphi^a = (\varphi^1, \varphi^2, \varphi^3)$.

The soliton solution of this model can be created with help of the complex variable $W = \dfrac{\varphi^1 + i\varphi^2}{1 - \varphi^3}$ using the north pole of stereographic projection from the sphere S^2 onto the plain $\mathbb{R}^2$. Then using a holomorphic map $W = \dfrac{P(z)}{Q(z)}$ where are polynomials of at most N of the complex coordinate $z = x + iy$ one can obtain N-soliton configuration of the field.

So let's start with the analysis of stereographic projection and then move on with our plan that we've discussed.

2. Stereographic projections

2.1. North projection

At first let's analyze the constrain on the field components that we have. The expression for the constrain is:

$$\varphi^a \cdot \varphi^a = \left(\varphi^1\right)^2 + \left(\varphi^2\right)^2 + \left(\varphi^3\right)^2 = 1$$

This expression determines a sphere S^2 of a unit radius with the center in the point of origin of a three dimensional field space. For the sphere itself, only two independent variables are required for its description so it's possible to decrease the dimension of the field space.

So let's create a one-to-one correspondence from the S^2 sphere to $\mathbb{R}^2$ plane. For that purpose we place center of a unit sphere in the point of origin and then draw a line which begins at the top (North pole) of our sphere and intersects a sphere and a plane. This line generates a one-to-one correspondence for every point on the sphere to a single point on the plane except for the point of North pole.

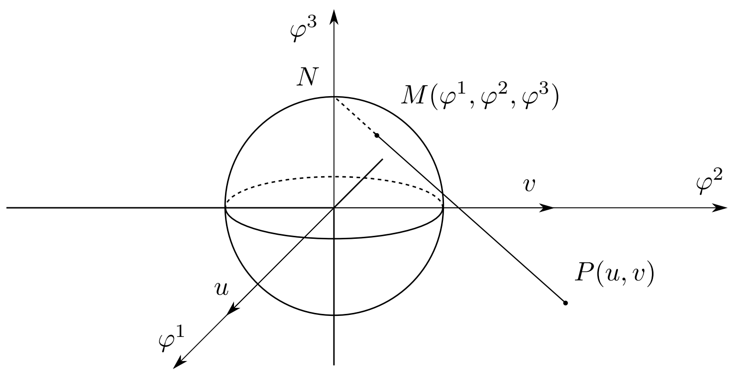


Figure 1: North projection

In 3-dimensional Cartesian coordinates an equation of a line that goes through two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Since our points are $N(0, 0, 1)$ and $M(\varphi^1, \varphi^2, \varphi^3)$, we obtain next formula:

$$\left.\frac{u}{\varphi^1} = \frac{w}{\varphi^2} = \frac{z-1}{\varphi^3 - 1}\right|_{z=0} \implies \frac{u}{\varphi^1} = \frac{w}{\varphi^2} = \frac{1}{1-\varphi^3}$$

$$\begin{cases} \dfrac{u}{\varphi^1} = \dfrac{1}{1-\varphi^3} \\ \dfrac{w}{\varphi^2} = \dfrac{1}{1-\varphi^3} \end{cases} \implies \boxed{(u, w) = \left(\frac{\varphi^1}{1-\varphi^3}, \frac{\varphi^2}{1-\varphi^3}\right)}$$

Now let's obtain inverse transformation:

$$\begin{cases} \dfrac{u}{\varphi^1} = \dfrac{1}{1-\varphi^3} \\ \dfrac{w}{\varphi^2} = \dfrac{1}{1-\varphi^3} \\ (\varphi^1)^2 + (\varphi^2)^2 + (\varphi^3)^2 = 1 \end{cases} \implies \begin{cases} \varphi^1 = u(1-\varphi^3) \\ \varphi^2 = w(1-\varphi^3) \\ (1-\varphi^3)^2 \cdot (1 + u^2 + w^2) = 2(1-\varphi^3) \end{cases}$$

And we gain inverse transformation by solving the system:

$$\boxed{(\varphi^1, \varphi^2, \varphi^3) = \left(\frac{2u}{1+u^2+w^2}, \frac{2w}{1+u^2+w^2}, -\frac{1-u^2-w^2}{1+u^2+w^2}\right)}$$

If we take one complex field variable $W = u + iw$ instead of two real field variables (u, w) we can introduce one-to-one correspondence from $(\varphi^1, \varphi^2, \varphi^3)$ to a field W and it's conjugate $\overline{W}$ which are not independent.

$$\begin{cases} u = \frac{1}{2}(W + \overline{W}) \\ w = -\frac{i}{2}(W - \overline{W}) \end{cases} \implies \boxed{(\varphi^1, \varphi^2, \varphi^3) = \left(\frac{W + \overline{W}}{1 + W\overline{W}},\ -i\frac{W - \overline{W}}{1 + W\overline{W}},\ -\frac{1 - W\overline{W}}{1 + W\overline{W}} \right)}$$

Note that:

$$1+(u^2+w^2) = 1+\frac{1}{4}\left(W^2 + \overline{W}^2 + 2W\overline{W} - W^2 - W^2 - \overline{W}^2 + 2W\overline{W}\right) = 1+(W\overline{W})$$

$$1 - (u^2 + w^2) = 1 - (W\overline{W})$$

2.2. South projection

If we take a South pole $S(0, 0, -1)$ as an origin of a line we will get almost identical expressions for field components by using the same algorithm. Let's show that and the difference that appears.

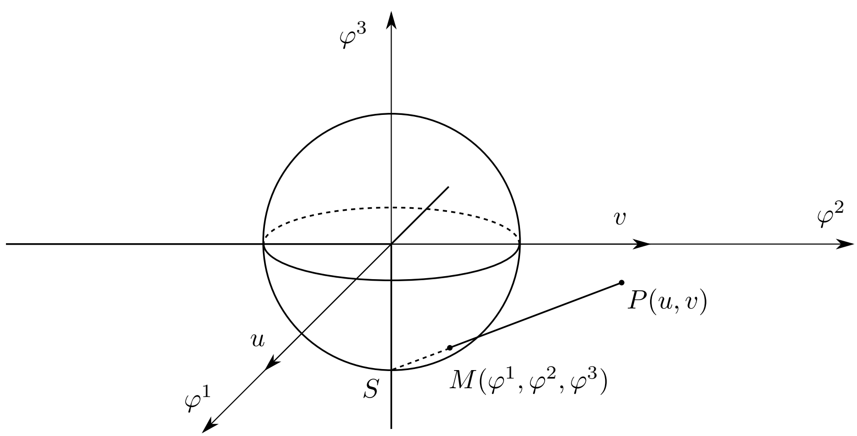


Figure 1: South projection

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Since our points are $S(0,0,-1)$ and $M(\varphi^1, \varphi^2, \varphi^3)$, we obtain next formula:

$$\left.\frac{u}{\varphi^1} = \frac{w}{\varphi^2} = \frac{z+1}{\varphi^3+1}\right|_{z=0} \Longrightarrow \frac{u}{\varphi^1} = \frac{w}{\varphi^2} = \frac{1}{1+\varphi^3}$$

$$\begin{cases} \dfrac{u}{\varphi^1} = \dfrac{1}{1+\varphi^3} \\ \dfrac{w}{\varphi^2} = \dfrac{1}{1+\varphi^3} \end{cases} \Longrightarrow \boxed{(u, w) = \left(\frac{\varphi^1}{1+\varphi^3}, \frac{\varphi^2}{1+\varphi^3}\right)}$$

And inverse transformation:

$$\begin{cases} \dfrac{u}{\varphi^1} = \dfrac{1}{1+\varphi^3} \\ \dfrac{w}{\varphi^2} = \dfrac{1}{1+\varphi^3} \\ (\varphi^1)^2 + (\varphi^2)^2 + (\varphi^3)^2 = 1 \end{cases} \Longrightarrow \begin{cases} \varphi^1 = u(1+\varphi^3) \\ \varphi^2 = w(1+\varphi^3) \\ (1+\varphi^3)^2 \cdot (1+u^2+w^2) = 2(1+\varphi^3) \end{cases}$$

$$\boxed{(\varphi^1, \varphi^2, \varphi^3) = \left(\frac{2u}{1+u^2+w^2}, \ \frac{2w}{1+u^2+w^2}, \ \frac{1-u^2-w^2}{1+u^2+w^2}\right)}$$

After introducing complex field $W = u + iw$ we get:

$$\begin{cases} u = \dfrac{1}{2}(W + \overline{W}) \\ w = -\dfrac{i}{2}(W - \overline{W}) \end{cases} \Longrightarrow \boxed{(\varphi^1, \varphi^2, \varphi^3) = \left(\frac{W+\overline{W}}{1+W\overline{\overline{W}}}, \ -i\frac{W-\overline{W}}{1+W\overline{\overline{W}}}, \ \frac{1-W\overline{\overline{W}}}{1+W\overline{\overline{W}}}\right)}$$

As we can see the expressions are almost identical except for the sign of the last potential φ^3.

2.3. Connection between projections

We have a triplet of potentials $(\varphi^1, \varphi^2, \varphi^3)$ that determines a field and its configuration. However this triplet has two different sets of representation using complex variable W which depend on the type of the projection. It means there exists a correspondence between these representations which we are to find now.

Let's consider a triplet that is produced by north projection and a complex variable W:

$$(\varphi^1, \varphi^2, \varphi^3) = \left(\frac{W + \overline{W}}{1 + W\overline{W}},\ -i\frac{W - \overline{W}}{1 + W\overline{W}},\ -\frac{1 - W\overline{W}}{1 + W\overline{W}} \right)$$

and a triplet produced by south projection with complex variable W':

$$(\varphi^{1\prime}, \varphi^{2\prime}, \varphi^{3\prime}) = \left(\frac{W' + \overline{W'}}{1 + W'\overline{W'}},\ -i\frac{W' - \overline{W'}}{1 + W'\overline{W'}},\ \frac{1 - W'\overline{W'}}{1 + W'\overline{W'}} \right)$$

Field potentials do not depend on the type of the projection, which means:

$$\begin{cases} \varphi^{1\prime} = \varphi^1 \\ \varphi^{2\prime} = \varphi^2 \\ \varphi^{3\prime} = \varphi^3 \end{cases} \Longrightarrow \begin{cases} \dfrac{W' + \overline{W'}}{1 + W'\overline{W'}} = \dfrac{W + \overline{W}}{1 + W\overline{W}} \\ -i\dfrac{W' - \overline{W'}}{1 + W'\overline{W'}} = -i\dfrac{W - \overline{W}}{1 + W\overline{W}} \\ \dfrac{1 - W'\overline{W'}}{1 + W'\overline{W'}} = -\dfrac{1 - W\overline{W}}{1 + W\overline{W}} \end{cases}$$

By solving this system of equations it can be shown that the connection between north and south projections has the following form:

$$W' = \frac{1}{\overline{W}}$$

Note that $W'' = W$ – every projection is identical to itself.

So now if we have complex field W that represents one of the projections of field potentials we can obtain complex field from the other projection W' which generates the same triplet of potentials and field configuration as well.

3. Construction of solitons

From previous part we got that our triplet $(\varphi^1, \varphi^2, \varphi^3)$ can be replaced by a single complex field variable $W = u + iw = \dfrac{\varphi^1 + i\varphi^2}{1 \pm \varphi^3}$ (the sign depends on the projection). This complex variable is determined at the every point of an (x, y) plane since we are in 2+1 dim model.

Every point of our plane can be mutually unambiguously mapped to a complex number $z = x + iy$. So our problem in creating a potentials $\varphi^a(x, y)$ is equivalent to a creating a function of complex variable $W(z)$ that can produce soliton solutions.

There is also a question about south and north projections. Which of those two projections should we use? Let's answer on that question.

3.1. Independence of energy

Despite the fact that for two different projections we have two different sets of transformations we can show that physical invariants of the fields (energy density, for example) remain the same.

Let's take a look at our Lagrangian density one more time:

$$\mathcal{L} = \frac{1}{4} \left(\partial_\mu \varphi^a \cdot \partial^\mu \varphi^a\right)$$

It's clear that our Lagrangian density does not depend on potentials themselves but only on their derivatives. Because of that its stress-energy tensor can be written as:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^a)} \partial_\nu \varphi^a - \eta^{\mu\nu} \mathcal{L}, \quad \eta = diag(-1, 1, 1)$$

We consider static configurations of the field, which leads to $\partial_0 \varphi^a = 0$. Thus energy density as a T^{00} component is:

$$\varepsilon = T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi^a)}\partial_0 \varphi^a - \eta^{00}\mathcal{L} = \mathcal{L}$$

Due to stationarity of our model Lagrangian density equals to:

$$\mathcal{L} = \frac{1}{4}\sum_{a=1}^{3}\left\{\left(\frac{\partial \varphi^a}{\partial x^1}\right)^2 + \left(\frac{\partial \varphi^a}{\partial x^2}\right)^2\right\}$$

As we remember the transformation laws are:

$$(\varphi^1, \varphi^2, \varphi^3) = \left(\frac{W + \overline{W}}{1 + W\overline{W}},\ -i\frac{W - \overline{W}}{1 + W\overline{W}},\ \pm\frac{1 - W\overline{W}}{1 + W\overline{W}}\right)$$

If we take a derivative ∂_μ of the potentials, we will obtain:

$$\begin{cases} \partial_\mu \varphi^1 = \dfrac{1}{(1 + W\overline{W})^2}\left(\partial_\mu(W + \overline{W}) + (\overline{W}^2\partial_\mu W + W^2\partial_\mu \overline{W})\right) \\ \partial_\mu \varphi^2 = \dfrac{-i}{(1 + W\overline{W})^2}\left(\partial_\mu(W - \overline{W}) + (\overline{W}^2\partial_\mu W - W^2\partial_\mu \overline{W})\right) \\ \partial_\mu \varphi^3 = \dfrac{\mp 2}{(1 + W\overline{W})^2}\left(W\partial_\mu \overline{W} + \overline{W}\partial_\mu W\right) \end{cases}$$

Now it can be seen that even if the sign of $\partial_\mu \varphi^3$ depends on the type of projection the energy density remains the same because only the squares of derivates are used in it. And there is nothing unusual in the expressions for $\partial_\mu \varphi^a$ that could affect this.

It can be shown [1] that if we use complex fields W and $\overline{W}$ the Lagrangian density (and energy density) will look like:

$$\mathcal{L} = \frac{\partial_\mu W \partial^\mu \overline{W}}{(1 + W\overline{W})^2}$$

And by introducing new derivatives Lagrangian density is changed to:

$$\begin{cases} \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \\ \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y) \end{cases} \implies \boxed{\mathcal{L} = \frac{|\partial_z W|^2 + |\partial_{\overline{z}} W|^2}{(1+|W|^2)^2}}$$

3.2. Rational mapping

Now as we have shown that the field invariants don't depend on the projections we can focus on creating a soliton solution for our model. Any soliton solution of our model can be constructed using rational mappings [1].

Rational mapping is a mapping:

$$W = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials of degree at most N.

3.3. Examples of solutions

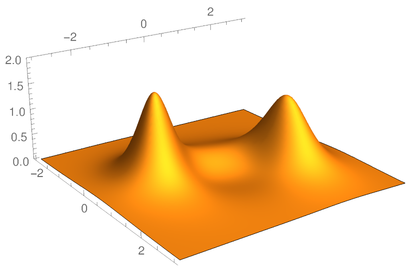
Now we have the method of constructing any N-soliton solution for our 2+1 dim O(3) sigma model. Let's have a look at examples of application of this method.

3.3.1 Two soliton solution

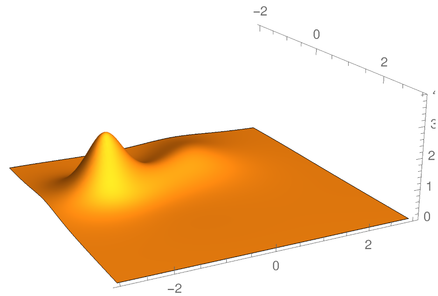
For two soliton solution we will consider following mapping:

$$W(z) = \frac{(z-a)(z-b)}{(z-c)(z-d)}$$

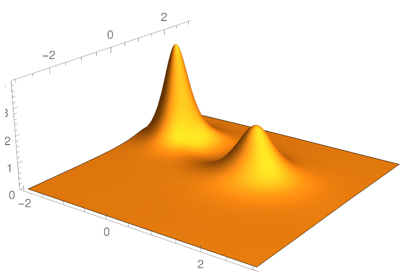
where a, b, c, d are complex numbers.



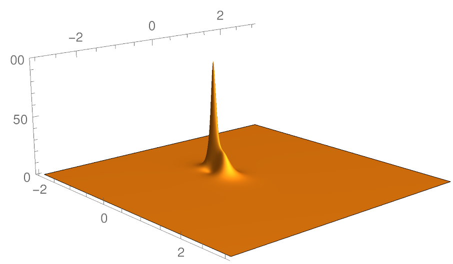
(a) Energy density for
$a = -0.08 - 1.9i$, $b = 0.79 + 2i$,
$c = 2 + 0.27i$, $d = 0.25 - 0.28i$



(b) Energy density for
$a = 0.75 - 1.9i$, $b = 0.1 + 1.37i$,
$c = -2.49 - 0.61i$, $d = -0.88 - 1.38i$



(c) Energy density for
$a = -1.55 + 1.27i$, $b = 1.53 + 0.37i$,
$c = -0.28 + 0.34i$, $d = -0.14 + 1.3i$



(d) Energy density for
$a = 0.24 + 0.105i$, $b = -0.21 - 0.03i$,
$c = 0.02 - 0.435i$, $d = -0.225 - 0.28i$

Figure 1: 4 distributions of energy density for different constants a, b, c, d

3.3.2 Eight solitons in a row

We can create a configuration of the field where 8 solitons are aligned along x-axis by using following map. For example using north projection we will have:

$$W(z) = \frac{1}{(z-x_1)(z-x_2)(z-x_3)(z-x_4)(z-x_5)(z-x_6)(z-x_7)(z-x_8)}$$

where $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ are real numbers that show the position of a soliton on the x-axis.

Equivalent mapping for the south projection is:

$$W'(z) = (\overline{z}-x_1)(\overline{z}-x_2)(\overline{z}-x_3)(\overline{z}-x_4)(\overline{z}-x_5)(\overline{z}-x_6)(\overline{z}-x_7)(\overline{z}-x_8)$$

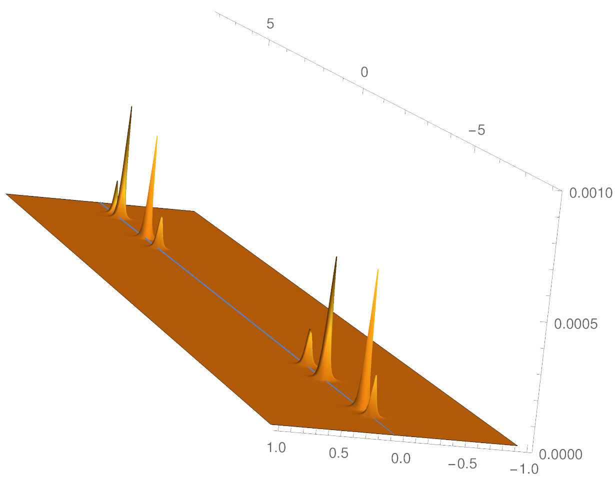


Figure 2: Energy density of 8 solitons along x-axis with
$x_1 = 7$, $x_2 = 6.6$, $x_3 = 4.9$, $x_4 = 4.1$, $x_5 = -4.1$, $x_6 = -4.9$, $x_7 = -6.6$, $x_8 = -7$

3.3.3 Eight solitons in a triangle

And in the end let's consider following mapping using north projection:

$$W = \frac{4}{\frac{1}{z} + \frac{1}{z+\frac{1}{2}-i} + \frac{1}{z-\frac{1}{2}-i} + \frac{1}{z-1} + \frac{1}{z+1} + \frac{1}{z+\frac{3}{2}+i} + \frac{1}{z-\frac{3}{2}+i} + \frac{1}{z-2i}}$$

For the south projection equivalent mapping is:

$$W'(z) = \frac{1}{4}\left(\frac{1}{\overline{z}} + \frac{1}{\overline{z}+\frac{1}{2}+i} + \frac{1}{\overline{z}-\frac{1}{2}+i} + \frac{1}{\overline{z}-1} + \frac{1}{\overline{z}+1} + \frac{1}{\overline{z}+\frac{3}{2}-i} + \frac{1}{\overline{z}-\frac{3}{2}-i} + \frac{1}{\overline{z}+2i}\right)$$

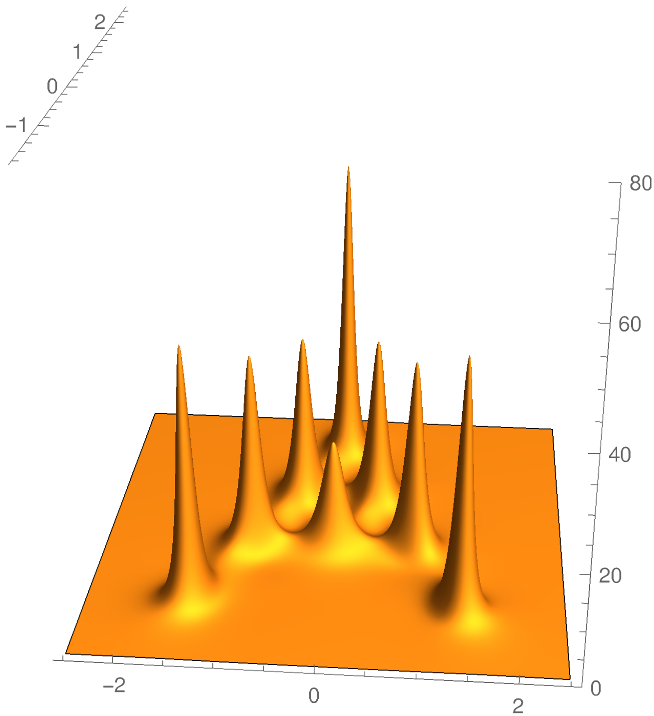
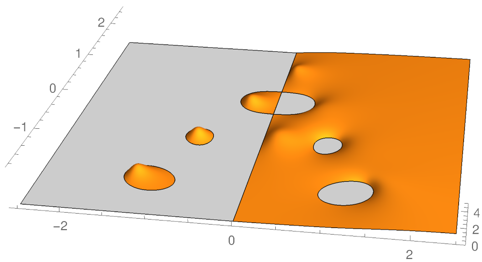
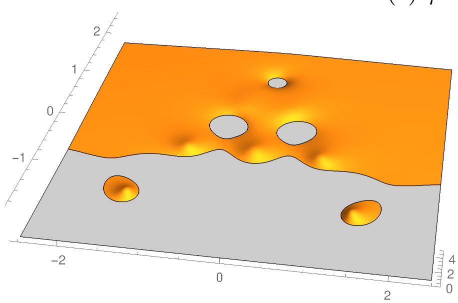


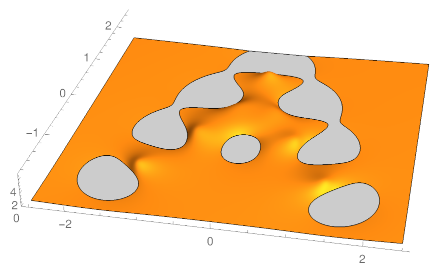
Figure 3: Energy density of 8 solitons generated by mapping in 3.3.3



(a) φ^1 potential



(b) φ^2 potential



(c) φ^3 potential

Figure 4: Three potentials $\varphi^1, \varphi^2, \varphi^3$ for the mapping given in 3.3.3

4. Conclusion

We have considered dim 2+1 O(3) sigma model.

To create a stationary soliton solutions we started with the analysis of stereographic projections and derived forward and inverse transformations. Two types of projections were considered – a projection which uses the North pole of potential sphere and another projection which uses South pole.

It was shown that field invariants (e.g. energy density) do not change whatever type of projection is used.

And then by using rational mappings we constructed several different many-soliton solutions. Various energy density and potential distributions were obtained. This was a demonstration of the fact that any N-soliton solution can be produced by this kind of a mapping.

5. References

[1] Y. M. Shnir. *Topological and non-topological solitons in scalar field theories.* Cambridge University Press, 2018.