JOINT INSTITUTE FOR NUCLEAR RESEARCH
Bogoliubov Laboratory of Theoretical Physics

FINAL REPORT ON INTEREST PROGRAMME

Numerical methods in theory of topological solitons

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Abstract

In this paper soliton solutions to particular systems are discussed, together with utilsed C++ solvers. Firstly the 3D Skyrme model is outlined and a spherically symmetric solitonic solution is presented. Subsequently the self-gravitating Skyrme model is set up, by coupling the classical version to Schwarzschild-like metric. Solution to this system is described with an analysis of a particular specimen. Relations to "No-Hair Conjecture" is discussed.

1 Introduction

Skyrme model was devised in the 1960s by T. Skyrme as an effective nucleon model. However, not being a valid renormalisable Quantum Field Theory model, it did not gain significant recognition. It was not until 1980s, when the theory received greater attention due to its capacity to produce solitonic solutions with interesting topological structure. Since then Skyrmions proved useful in such areas as condensed matter, and, notably for this paper, gravitational physics [1]. I particular self-gravitating Skyrmions were found counterexemplary to so-called black hole no hair conjecture [4], as discussed in upcoming sections.

2 Classical Skyrme Model

2.1 Theoretical Background

The model emerges from a $SU(2)\times SU(2) \simeq SO(4)$-valued chiral field, but chiral symmetry may be broken with field simplified to $U = SU(2)$. The resulting Lagrangian becomes [1]:

$$\int \frac{F^2}{16} tr(U^\dagger \partial_\mu U U^\dagger \partial_\mu U) + \frac{1}{32e^2} tr[\partial_\mu U U^\dagger, \partial_\nu U U^\dagger]^2 \, d^4x \tag{1}$$

with μ, ν running between 0 and 4, 0 corresponding to time derivative, dagger indicates hermitian conjugate, $[,]$ is the Lie bracket, and F, e are coupling constants. Note that the Lagrangian in this form is Lorentz covariant. However, by Principle of Symmetric Criticality, it is possible to reduce the number of coordinates involved using the structure of $SU(2)$. Namely a hedgehog ansatz can be invoked, assuming spherically symmetric profile $f(r)$ for the arising quartet of fields $(\sigma, \phi_1, \phi_2, \phi_3)$. The principle guarantees that if such solution can be found for the symmetry-broken subgroup, it is valid for the whole model. The ansatz in question is

$$\phi_k = \exp(if(r)\boldsymbol{\tau_k} \cdot \hat{\boldsymbol{r}}) \tag{2}$$

with $\hat{\boldsymbol{r}}$ being the radial vector pointing outwards, k running over space coordinates and $\boldsymbol{\tau_k}$ being corresponding Pauli matrices. The boundary conditions for the profile function are

$$f(0) = \pi, f(\infty) = 0 \tag{3}$$

Under this transformation, after applying Euler-Lagrange equations to eq. (1), the resulting field equation is

$$(r^2 + 2\sin^2 f)f'' + 2rf' - \sin 2f(1 - (f')^2 + \frac{\sin^2 f}{r^2} + m^2 \sin f = 0 \quad (4)$$

where differentiation is with respect to r and $m = \frac{m_p i}{F}$ is the dimensionless pion mass. Equation (4), can be then solved numerically.

2.2 Solution

Equation (4) was solved using finite difference method, implemented using standard C++ libraries [3]. In particular the derivatives were approximated using discretised central difference approximations

$$f'_{i+1} = \frac{f_{i+2} - f_i}{2\delta}, f''_{i+1} = \frac{f_{i+2} - 2f_{i+1} + f_i}{\delta^2}, i = 1, ..., N \quad (5)$$

where δ is the step in r mesh. Applying (5) to (4) and (3), results in a system of N nonlinear equations of form as in (6).

$$m^2 \sin(f_{i+1}) - (\frac{(1 + \sin(f_{i+1})^2}{r^2} - \frac{(-f_{i+1} + f_{i+2})^2}{\delta^2}) \sin(2f_{i+1}) + \frac{2r(-f_{i+1} + f_{i+2})}{\delta} + \\ + (r^2 + 2\sin(f_{i+1})^2) \frac{(f_i - 2f_{i+1} + f_{i+2})}{\delta^2} = 0 \quad (6)$$

with $f_1 = \pi$ and $f_N = 0$ (with a large finite number rather than infinity for the right boundary).
Due to non-linearity of (6), in order to be solved the state vector had to be approximated using the vectorial Raphson-Newton algorithm. Namely, if $\boldsymbol{F}^k$ is the vector of functions $f_1^k, ..., f_N^k$ on the grid in the kth step of Raphson-Newton algorithm, and $E_1, .., E_N$ are the right-hand sides of corresponding equations (6), then:

$$J(\boldsymbol{F}^k)\Delta\boldsymbol{F}^k = -\boldsymbol{F}^k \quad (7)$$

where $\Delta\boldsymbol{F}^k = \boldsymbol{F}^{k+1} - \boldsymbol{F}^k$ and $J(\boldsymbol{F}^k)$ is the Jacobian matrix given by

$$J(\boldsymbol{F}^k) = \begin{bmatrix} \frac{\partial E_1}{\partial f_1^k} & ... & \frac{\partial E_1}{\partial f_N^k} \\ ... & ... & ... \\ \frac{\partial E_N}{\partial f_1^k} & ... & \frac{\partial E_k}{\partial f_N^k} \end{bmatrix}. \quad (8)$$

Note that Jacobian matrix is tri-diagonal, so (7) can be inverted using Thomas algorithm, again easily implementable in C++ setup, yielding sought result. Raphson-Newton method requires an initial guess, which was chosen according to [1]

$$\boldsymbol{F}^1 = \begin{bmatrix} \frac{\pi}{(r_1)^2} \\ ... \\ \frac{\pi}{(r_N)^2} \end{bmatrix} \quad (9)$$

2.3 Results

The scheme was run for several values of parameter pion mass, with $N = 60, \delta = 0.1$ and using 10 iterations of (7) to ensure convergence.The obtained radial profile can be seen in fig. 1. The profile agrees with the one found in [1]. Its

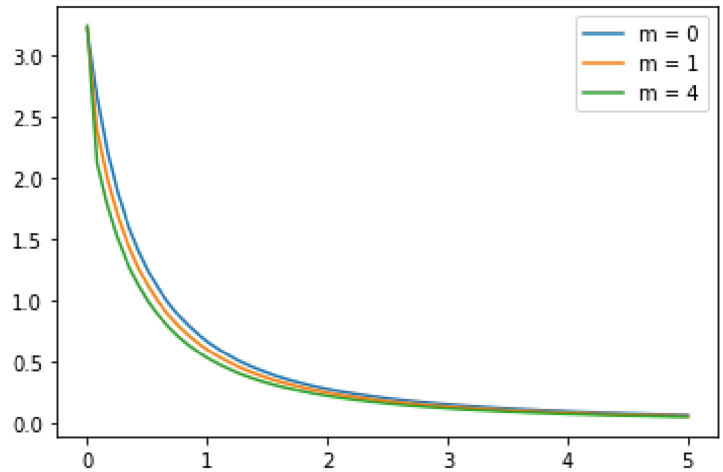

Figure 1: Profile function of hedgehog ansatz in classical Skyrme model. Notice the accordance with $1/r^2$ regime. The steepness of descent appears to increase with increasing pion mass m.

resemblance to Coulomb-like inverse square law is justified, as the obtained in eq. (2) pion fields form a set of mutually orthgonal dipoles [2].

3 Self-Gravitating Skyrme Model

3.1 Theoretical Background

In order to allow the Skyrmion to gravitate, Lagrangian (1) need be coupled to a spacetime metric [4]. To remain in the spherically symmetric regime, a modified Schwarzschild Metric in spherical polar coordinates (physics convention) can be consider as in (10)

$$ds^2 = -A^2(r)(1 - \frac{2m(r)}{r})dt^2 + (1 - \frac{2m(r)}{r})^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin\theta d\phi^2 \quad (10)$$

where $m(r)$ is a radial mass function and, $A(r)$ - a dimensionless radial profile function. Let also ∇_ν be the associated covariant derivative with index ν running conventionally as in (1). Using this metric, the initial Lagrangian (eq. (1)), being already Lorentz covariant may be transformed to

$$\int \frac{F^2}{4} tr(\nabla_\mu U \nabla^\mu U^{-1}) + \frac{1}{32e^2} tr[(\nabla_\mu U)U^{-1}, (\nabla_\nu U)U^{-1}]d^4x \quad (11)$$

utilising the same ansatz (eq. (2)) to eq. (11), and applying Euler-Lagrange equations, renders the system of three coupled field equations [4]

$$\mu' = \alpha[\frac{1}{2}x^2 N(f')^2 + \sin^2 f + \sin^2 f(N(f')^2 + \frac{\sin^2 f}{2x^2})] \tag{12}$$

$$[(x^2 + 2\sin^2 f)ANf']' = A\sin 2f(1 + N(f')^2 + \frac{\sin^2 f}{x^2}) \tag{13}$$

$$A' = \alpha(x + \frac{2}{x}\sin^2 f)A(f')^2 \tag{14}$$

where $x = eFr$, $\mu = efm(r)$ are scaled to coupling constant units, $\alpha = 4\pi GF$ is the effective gravity coupling constant (G is the Newton's constant), and $N = (1 - \frac{m(r)}{r})$ was introduced for brevity. Additionally $'$ indicates differentiation with respect to x. The boundary conditions for eqs. (12) - (14) read as in (16) [4]

$$f(0) = \pi, f(\infty) = 0 \tag{15}$$

$$\mu(0) = \tilde{M}_{BH}, \mu(\infty) = \tilde{M}_{Sol} \tag{16}$$

with $\tilde{M}_{BH}, \tilde{M}_{Sol}$ being mass of the black hole and soliton respectively, in the units of μ. This system was also solved numerically, this time using different approach.

3.2 Solution

Due to coupling of the system (12)-(14), Raphson-Newton method would become increasingly complicated and so a simpler shooting method was used instead. In this approach, this boundary value problem is transformed into an initial value problem with A, μ and f' being adjusted until the solution attains required values at infinity. The actual solving of the IVP was done using the ODEINT C++ library [5], with an adaptive Runge-Kutta 5th order stepper over the same r mesh as in previous solution (section 2.2). It has to be regrettably noted that due to limited computing capacity the full solution in [4], could not be reproduced. Instead one particular specimen will be analysed.

3.3 Result

Consider a case where $\alpha = 0.01$, $\mu(0) = 56.70$, $A(0) = -0.5$ and $f'(0) = 0.01$. The profile function obtained can be seen in 2.

This resulted in $\mu(\infty)$ tending to around 55.82, as can be seen in fig. An interesting result is the function A (fig. 4), corresponding for the coupling of the metric to the Skyrme model. It is asymptotically increasing in modulus meaning that the acquired solution possesses so-called Skyrme hair [4]. The soliton, having finite mass at infinity, interacts gravitationally in the far field, as opposed to the classical Skyrme model and regular Schwarzschild black hole, which both decay.

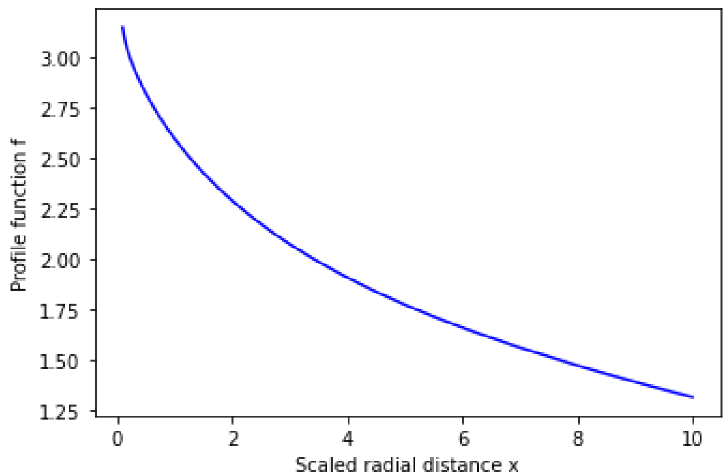

Figure 2: Profile function of the acquired gravitating soliton solution. Notice smaller steepness in comparison to fig. 1

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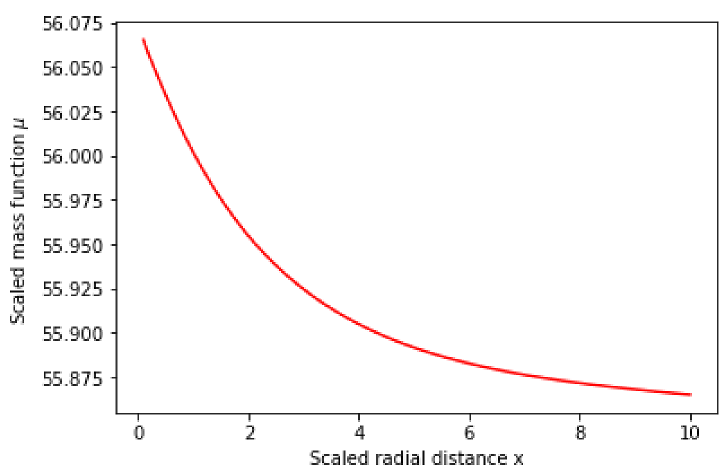

Figure 3: Scaled mass function of the acquired solution. Notice the assymptotic behaviour tending to a finite mass of the Skyrmion

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4 Summary

The classical and self-gravitating Skyrme models were outlined and compared. Solitonic solutions were obtained for each, bearing both similar and distinct features, particularly in the far field behaviour. It was noted that solving such

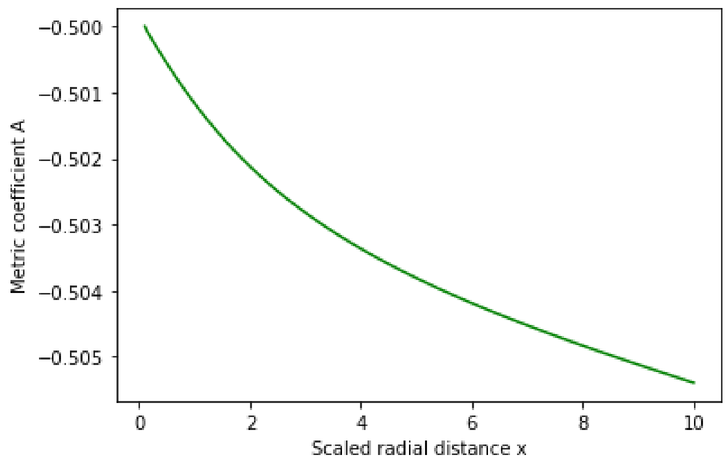

Figure 4: Metric coefficient function for the acquired solution. Notice the modulus increasing towards infinity indicating gravitational interaction of Skyrme hair in the far field

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systems numerically, even with a language as fast as C++, requires significant computing capacity. In the broader context, results of section 3 mathematically challenge the "no-hair conjecture". This postulate conceived in 1960s requires black hole solutions to be describable only by its mass, total charge and angular momentum, thus being compact and "hairless". Multiple systems, including Schwarzschild spacetime, were proved to either behave like so, or quickly decay from a unstable state to compact one. On the contrary, self-gravitating Skyrmion, by virtue of admitting topological solition solutions, is able to sustain stable hairy black holes. However, no physical process is known to give rise to such phenomena and thus their existnce does not appear to violate the conjecture in nature, for the present.

5 References

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