Student's report

Numerical methods in theory of topological soliton

Prepared by:
Smetankin Sergey

Supervisor:
Prof. Yakov Shnir

Contents

Abstract

In this project we consider kink (antikink) solutions for three model of field: ϕ^4, ϕ^6 and Sine Gordon. We use numerical methods to find static solutions and moving kink solution for ϕ^4 theory.

1 First view

Follow the Wikipedia a **soliton** or **solitary wave** is a self-reinforcing wave packet that maintains its shape while it propagates at a constant velocity. This waves are ubiquitous in nature and have many applications in nonlinear dynamics.

2 General theory

The most elementary topological solitons arise in 1+1 dimensional space that involve a single scalar field. An general Lagrangian of such scalar field can be written as,

$$L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - U \tag{1}$$

which lead to the corresponding field equation

$$\partial_\mu \partial^\mu \phi + U' = 0 \tag{2}$$

in Minkovsky signature

$$\partial^2_{tt}\phi - \partial^2_{xx}\phi + U' = 0 \tag{3}$$

for the static solution we consider

$$\partial^2_{xx}\phi - U' = 0 \tag{4}$$

with boundary conditions corresponding vacua

$$\phi(-\infty) = \pm 1, \;\; \phi(+\infty) = \mp 1 \tag{5}$$

3 ϕ^4model

$$U = \frac{1}{2}(1 - \phi^2)^2 \tag{6}$$

this model has two vacua

$$\phi(-\infty) = \pm 1, \;\; \phi(+\infty) = \mp 1 \tag{7}$$

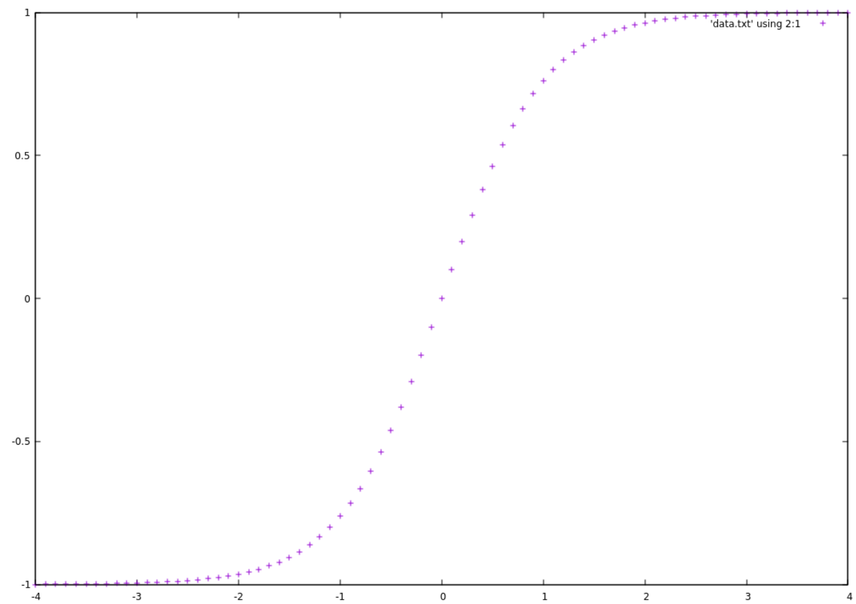
for kink solution we give

$$\phi(-\infty) = -1, \;\; \phi(+\infty) = 1 \tag{8}$$

solution is

$$\phi = \tanh(x - x_0) \tag{9}$$

but we found it numerically, using Newton method



For moving kink analitical solution we make some calculus

$$\phi_{tt} - \phi_{xx} + 2\phi(\phi^2 - 1) = 0 \tag{10}$$

$$\phi(x,t) = \phi(x - vt) = \phi(z) \tag{11}$$

$$(v^2 - 1)\phi" + 2\phi(\phi^2 - 1) = 0 | \cdot \phi' \tag{12}$$

$$(v^2 - 1)\phi'\phi" + 2\phi\phi'(\phi^2 - 1) = 0 \tag{13}$$

$$(v^2 - 1)\phi'^2 \frac{1}{2} + \frac{1}{2}(\phi^2 - 1)^2 + C = 0 \tag{14}$$

Using boundary conditions

$$\phi(\pm\infty) = \pm 1, \;\; \phi'(\infty) = 0 \Rightarrow C = 0 \tag{15}$$

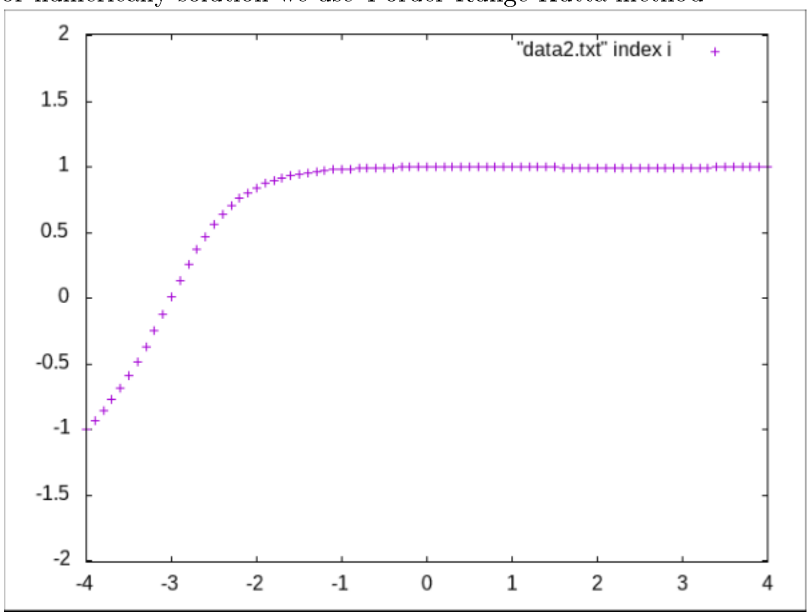
$$(v^2 - 1)\phi'^2 + (\phi^2 - 1)^2 = 0 \tag{16}$$

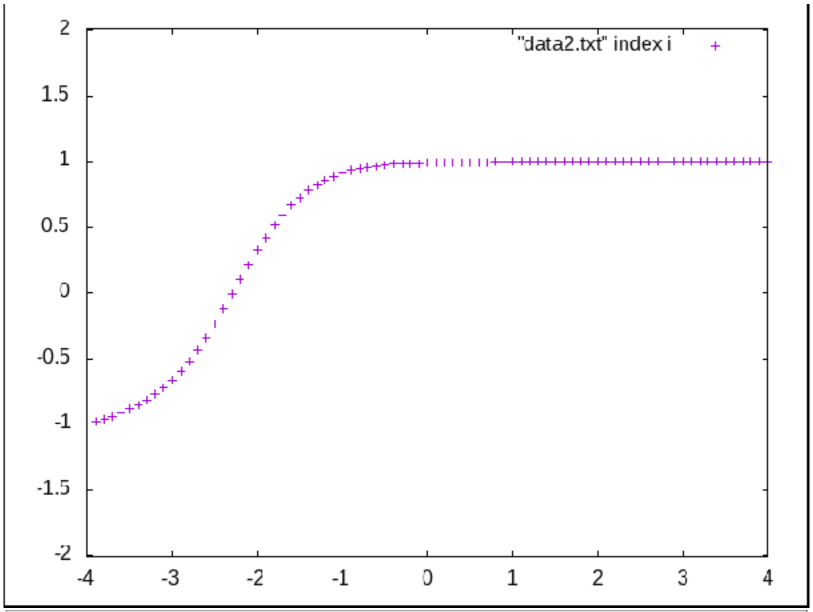
$$(1-v^2)\phi'^2 = (\phi^2-1)^2 \tag{17}$$

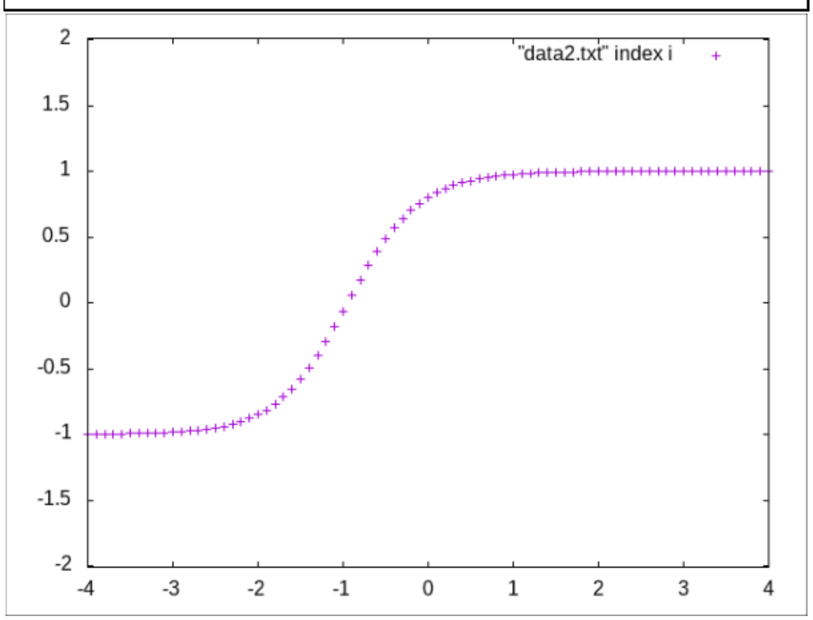
$$\phi' = \pm\frac{(\phi^2-1)}{\sqrt{(1-v^2)}} \tag{18}$$

$$\phi = th\left(\mp\frac{(x-x_0)-v(t-t_0)}{\sqrt{(1-v^2)}}\right) \tag{19}$$

For numerically solution we use 4 order Runge-Kutta method







4 ϕ^6model.

There the potential is

$$U = \frac{1}{2}\phi^2(1-\phi^2)^2 \tag{20}$$

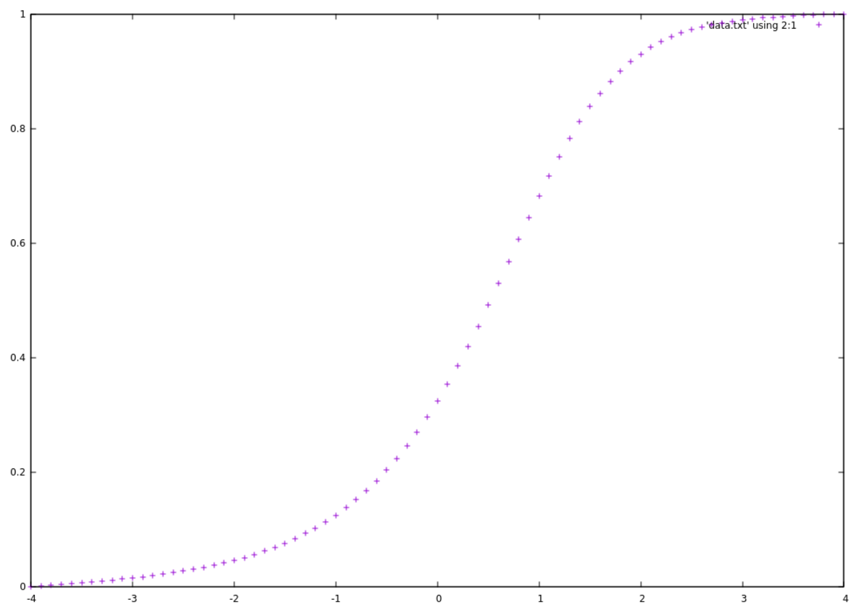
and vacua give following boundaries

$$\phi(-\infty) = 0, \;\; \phi(+\infty) = 1 \tag{21}$$

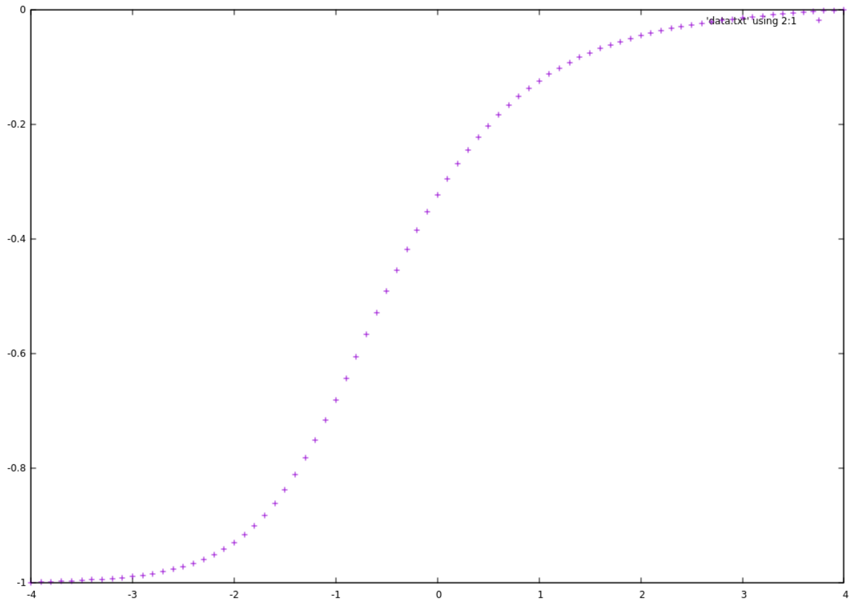
$$\phi(-\infty) = -1, \;\; \phi(+\infty) = 0 \tag{22}$$

analitical solution is

$$\phi = \pm \frac{1}{\sqrt{1 + e^{\mp 2(x - x_0)}}} \tag{23}$$



for boundaries 21



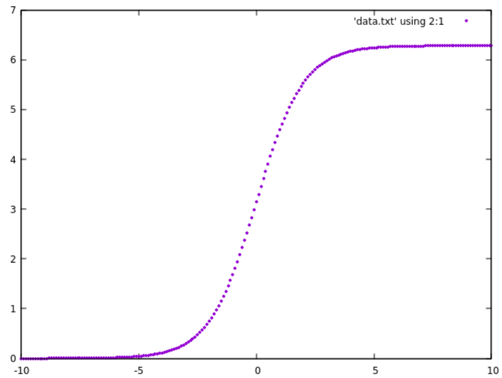
for boudaries 22

5 Sine-Gordon model

$$U = 1 - \cos(\phi)$$

$$\phi(-\infty) = 0, \;\; \phi(+\infty) = 2\pi$$

numerically solution is

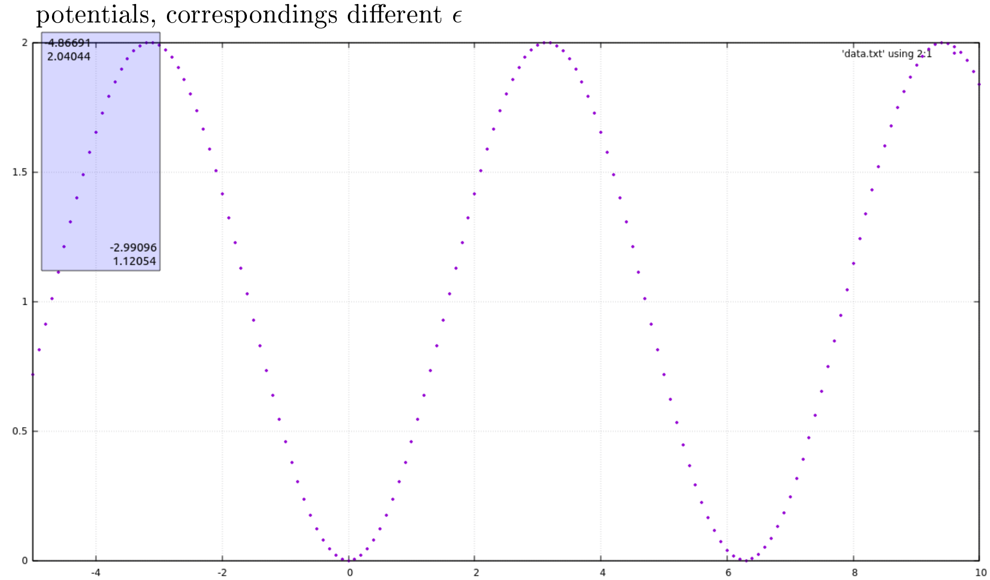


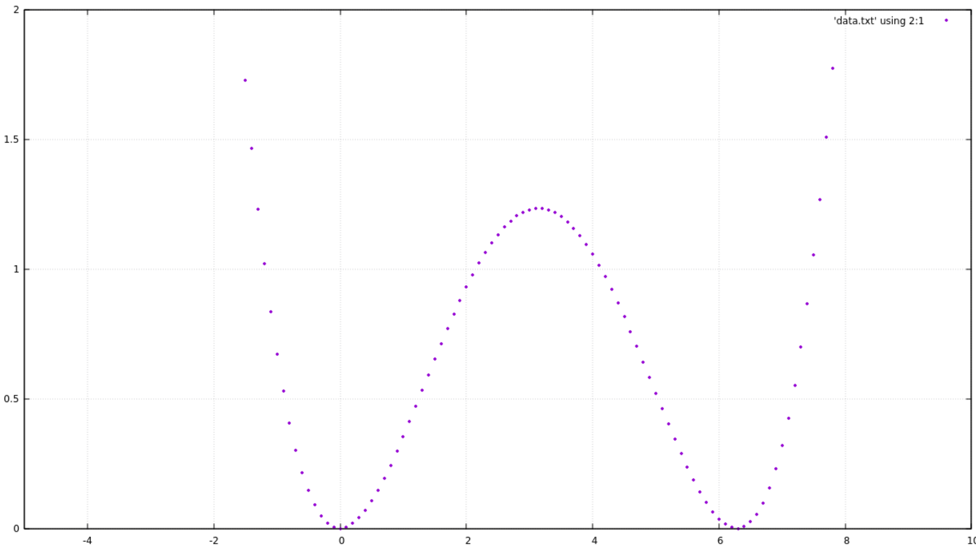
6 Parametrisesd model

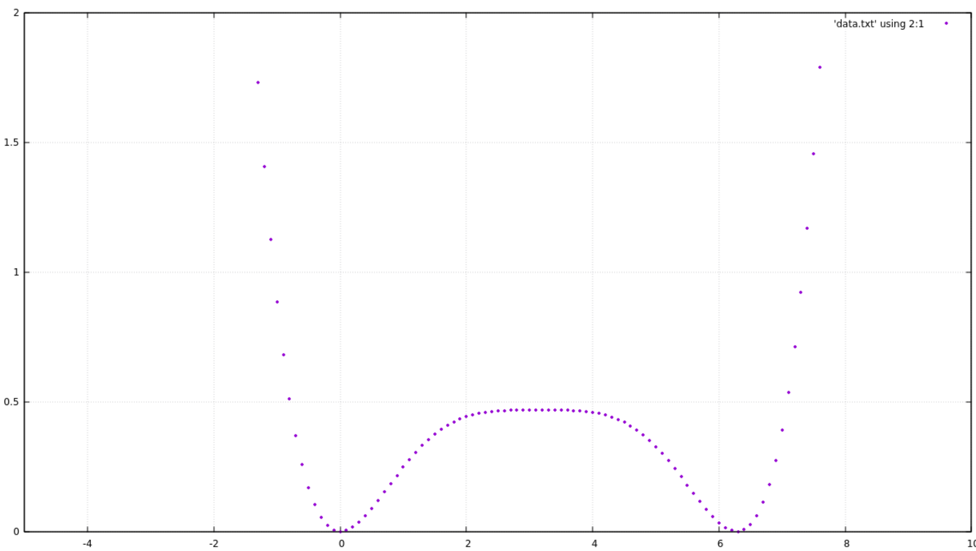
$$U = (1-\epsilon)(1-\cos(\phi)) + \frac{\epsilon\phi^2}{8\pi^2}(\phi - 2\pi)^2$$

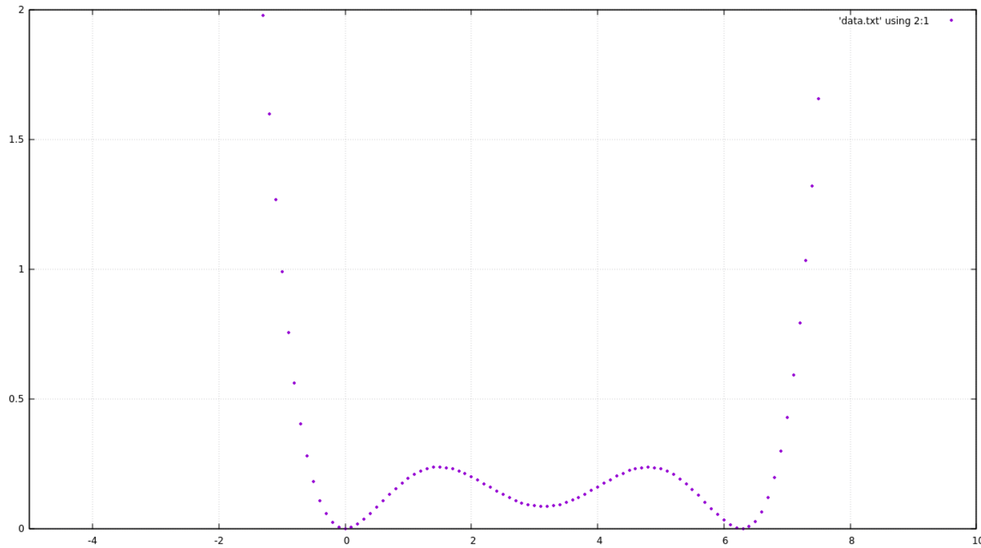
$$\phi(-\infty) = -1, \;\; \phi(+\infty) = 1$$

potentials, correspondings different ϵ

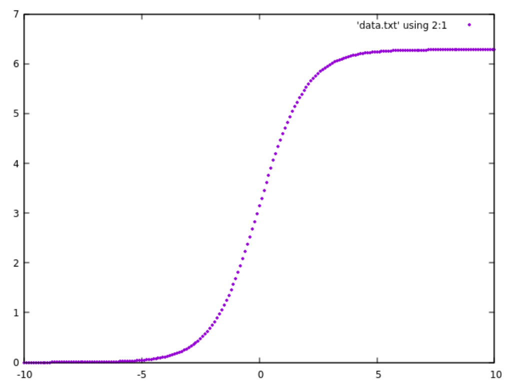


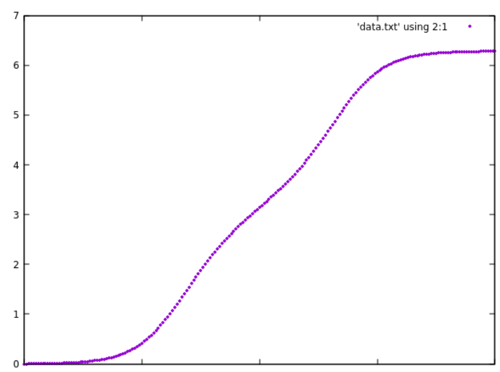

'data.txt' using 2:1

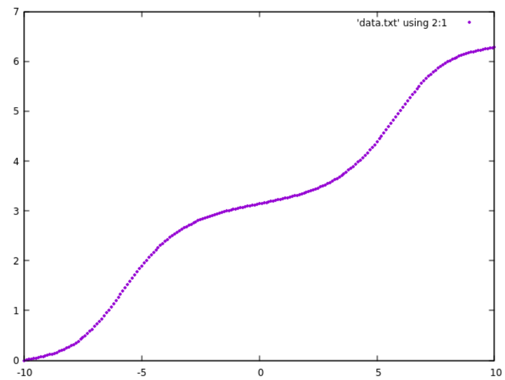

'data.txt' using 2:1



solutions for different potentials






'data.txt' using 2:1
0
1
2
3
4
5
6
7
-10
-5
0
5
10

7 References

1. Y. M. Shnir. Topological and non-topological solitons in scalar field theories. Cam- bridge University Press, 2018

2. W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. Numerical recipes 3rd edition: The art of scientific computing. Cambridge university press, 2007