JOINT INSTITUTE FOR NUCLEAR RESEARCH
Bogoliubov Laboratory of Theoretical Physics

FINAL REPORT ON THE INTEREST PROGRAMME

Energy loss in a Quantum Chromodynamic plasma

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Abstract

Energy loss in a Quantum Chromodynamics (QCD) plasma is currently an area of active research, as it helps us understand the underlying principles of QCD, which governs the behavior of quarks and gluons and the fundamental properties of the early universe.

In this project, numerical estimates of the energy loss of partons (quarks and gluons) due to elastic and inelastic or radiative processes in a QCD plasma are performed.

Contents

1 Introduction

In this report the vacuum speed of light (c), the reduced Planck constant ($\hbar$) and the Boltzmann constant (k_B) are set to 1. In these units, both length and time are expressed in inverse energy units.

1.1 Basics of QGP

Quark-gluon plasma (QGP) is a deconfined state of quarks and gluons. In a plasma the number of particles within the Debye screening length (λ_D): $n\lambda_{\mathrm{D}}^3 \gg 1$, where n is the particle number density.

In normal conditions, quarks are confined within hadrons due to the strong force, which acts as a color potential. However, when the energy density or temperature reaches a critical point ($\epsilon_{\mathrm{cr}} \approx \frac{1\ \mathrm{Gev}}{\mathrm{fm}^3}$; $T_{\mathrm{cr}} \approx 170$ MeV), the distance between two hadrons can be smaller than the hadron diameter, and quarks are no longer confined within individual hadrons [1]. Instead, they become deconfined and can move freely in the quark-gluon plasma, which is a state of matter that existed in the early universe just after the Big Bang or can be created in high-energy collisions experiments [2].

1.2 Evolution of QGP

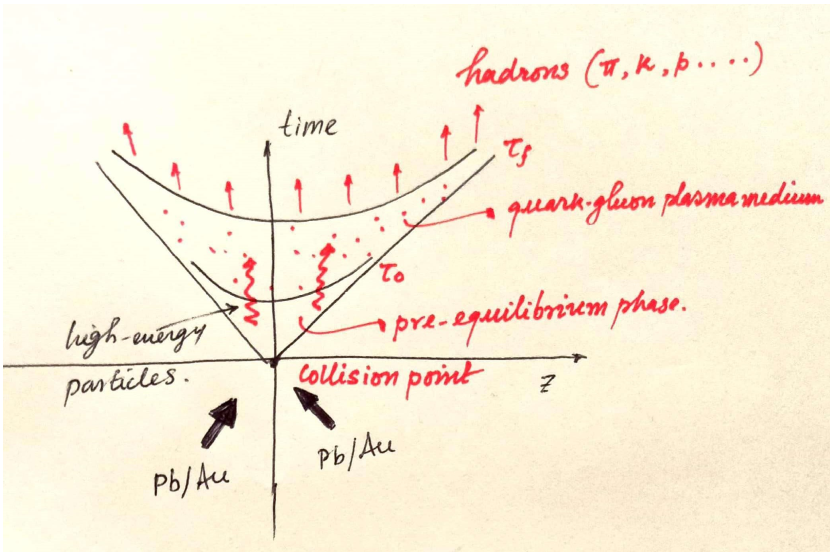

Figure 1: Space-time evolution of a heavy ion collision [3].

This space-time diagram shows the evolution of high-energy particles and QGP after the collision of heavy ions like Pb/Au. High-energy particles (e.g., light quarks, heavy quarks, and gluons) are created during the pre-equilibrium phase. It is expected that after a proper time $\tau_0 \approx 1$ fm of the collision, the QGP medium enters a state of local thermal equilibrium. Some of high-energy particles may pass through the medium and change their distribution function. The ratio of the outgoing to the initial distribution can be connected to the nuclear suppression factor R_{AA}, which is related to experimental observables in high-energy collision physics and gives information about the medium.

At some point in time (about $\tau_f \approx 10$ fm and $T_f \approx 120$ MeV) the expansion rate is so high that it does not let particles interact. So, particles stop exchanging momentum. That is called the kinetic freeze-out, and T_f is called the kinetic freeze-out temperature. At the freeze-out hyper-surface, the quarks and gluons undergo hadronization.

1.3 Evolution equations

We have two types of evolution in QGP:

1. evolution of the medium which is dictated by hydrodynamics equation: $\partial_\mu T^{\mu\nu} = 0$, where $T^{\mu\nu}$ is a stress-energy tensor;

2. evolution of energetic particles which is dictated by the Boltzmann transport equation.

Our project considers the second type of evolution.

1.4 Project goals

We aim to estimate the radiative energy loss of gluons inside a gluonic plasma and the energy loss of heavy quarks through collisional processes in a QGP. This will help us better understand the dynamics and properties of the QCD plasma.

2 Passage of energetic particles in the QCD plasma

In the high-energy regime, where QGP is formed, the gluonic distribution dominates over the quark distribution. So, it is common to approximate QGP as a gluonic plasma, neglecting the effects of quark dynamics, at extremely high temperatures.

Among various inelastic processes involving quarks and gluons, the process: $g(k_1) + g(k_2) \to g(k_3) + g(k_4) + g(k_5)$ plays a major role in the system's equilibration.

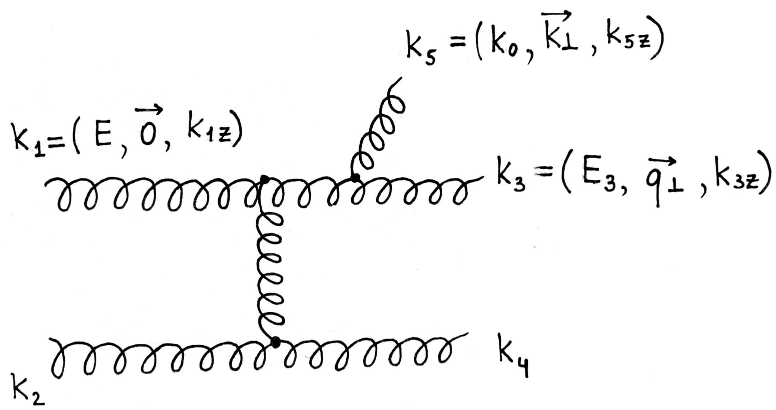

Figure 2: One of the Feynman diagrams for $g(k_1) + g(k_2) \to g(k_3) + g(k_4) + g(k_5)$ with notations used below. Assuming that the incoming gluons have no transverse momentum, i.e. they are traveling along the z-axis.

2.1 Radiative energy loss in a gluonic plasma

The energy loss per collision can be estimated by multiplying the probability of radiating a gluon and the energy of the gluon [4]:

$$\epsilon = \int \underbrace{d^2k_\perp d\eta \frac{dn_g}{d^2k_\perp d\eta}}_{\text{Number of gluons emitted}} \times \underbrace{k_0}_{\text{Energy carried by each gluon}} \times \underbrace{\theta(\Lambda^{-1} - \tau_F)\theta(E - k_0)}_{\text{Constraints}} =$$

$$= \int_{k_\perp = \Lambda \cosh\eta}^{k_\perp = \frac{E}{\cosh\eta}} d^2k_\perp d\eta \frac{dn_g}{d^2k_\perp d\eta} \times k_\perp \cosh\eta, \tag{1}$$

where $\vec{k}_\perp$ is the transverse momentum of the emitted gluon, $\eta = \frac{1}{2}\ln\frac{k_0+k_{5z}}{k_0-k_{5z}}$ is the rapidity of the emitted gluon, n_g is the number of emitted gluons,

$k_0 = k_\perp \cosh\eta$ is the energy of the emitted gluon,[1] θ is the Heaviside step function, Λ^{-1} is the interaction time, $\tau_F \approx \frac{\cosh\eta}{k_\perp}$ is the formation time of the gluon,[2] and E is the energy of the incoming high-energy gluon (the upper limit for the energy of the emitted gluon). The first θ-funstion sets that the effective formation time, τ_F must be smaller than the interaction time, Λ^{-1}.[3]

If the transverse momentum transfer, $q_\perp$ is much greater than $k_\perp$, the spectrum of the emitted gluon from the $gg \to ggg$ process is given by

$$\left.\frac{dn_g}{d^2k_\perp d\eta}\right|_{q_\perp \gg k_\perp} = \frac{C_A\alpha_s}{\pi^2}\frac{1}{k_\perp^2}, \tag{2}$$

where $C_A = 3$ is the Casimir invariant for the SU(3) adjoint representation, and α_s is the temperature-dependent strong coupling constant. We can replace $d^2k_\perp$ with $k_\perp dk_\perp d\theta$ in Eq. (1). ϵ is then obtained as

$$\begin{aligned}
\epsilon|_{q_\perp \gg k_\perp} &= \frac{C_A\alpha_s}{\pi^2} \times \int_{\Lambda\cosh\eta}^{\frac{E}{\cosh\eta}} \int_0^{\pi} \int_{-0.5}^{0.5} dk_\perp d\theta d\eta \times \frac{1}{\cancel{k_\perp^2}} \times \cancel{k_\perp}\cosh\eta = \\
&= \frac{C_A\alpha_s}{\pi^2} \times \int_0^{\pi} d\theta \times \int_{-0.5}^{0.5} d\eta \left(\cosh\eta \int_{\Lambda\cosh\eta}^{\frac{E}{\cosh\eta}} dk_\perp\right) = \\
&= \frac{C_A\alpha_s}{\pi} \times \int_{-0.5}^{0.5} d\eta\,(E - \Lambda\cosh\eta) \approx \alpha_s \times (0.95 \times E - 1.00 \times \Lambda).
\end{aligned} \tag{3}$$

For $\alpha_s = 0.3$ and $\Lambda = 0.3$ GeV,[4] the energy loss per collision for a $E = 5$ GeV gluon is approximately 1.3 GeV. From the analytical form of ϵ it can be seen that *increasing the strong coupling (α_s), the interaction time (Λ^{-1}) or the energy of the incoming high-energy gluon (E) results in an increase the energy loss per collision.*

If $k_\perp \gtrsim q_\perp$, the spectrum of the emitted gluon in the medium can be given by Gunion-Bertsch formula [7]:

$$\frac{dn_g}{d^2k_\perp d\eta} = \frac{C_A\alpha_s}{\pi^2}\frac{q_\perp^2}{k_\perp^2[(\vec{k}_\perp - \vec{q}_\perp)^2 + m_D^2]} = \tag{4}$$

$$= \frac{C_A\alpha_s}{\pi^2}\frac{q_\perp^2}{k_\perp^2[\omega^2 + a^2]}, \tag{5}$$

[1] We can parameterize k_0 and k_{5z} in terms of $k_\perp$ and η due to masslessness of gluons: $k_0^2 + k_{5z}^2 = k_\perp^2$.

[2] The formation time for gluons is $\tau_F = \frac{1}{2}\tau_{\mathrm{QED}} \approx \frac{1}{2}\frac{2k_0}{k_\perp^2}$ [5]. For calculating the formation time in QED, τ_{QED}, see Appendix A.

[3] In this limit, the intensity of induced radiation is additive in the number of scatterings.

[4] In this report we consider $\Lambda \approx T$. For more rigorous estimates, see [6].

where $\omega = k_\perp - q_\perp \cos\theta$, $a^2 = q_\perp^2 \sin^2\theta + m_D^2$, $\theta = \angle(\vec{k}_\perp, \vec{q}_\perp)$, $m_D \approx \sqrt{4\pi\alpha_s}T$ is the thermal (Debye) mass of the gluon required to shield the infrared divergence,[5] and T is the temperature of the medium. We can replace $d^2k_\perp$ with $k_\perp d\omega d\theta$ in Eq. (1) and use the average values of $q_\perp$ and $q_\perp^2$:

$$\langle q_\perp \rangle = \frac{1}{\sigma_{el}} \int_{m_D^2}^{\frac{s}{4}} dq_\perp^2 \frac{d\sigma_{el}}{dq_\perp^2} q_\perp = \frac{2\sqrt{s}m_D}{\sqrt{s} + 2m_D},$$
$$\langle q_\perp^2 \rangle = \frac{1}{\sigma_{el}} \int_{m_D^2}^{\frac{s}{4}} dq_\perp^2 \frac{d\sigma_{el}}{dq_\perp^2} q_\perp^2 = \frac{s m_D^2}{s - 4m_D^2} \ln\left(\frac{s}{4m_D^2}\right), \tag{6}$$

where $s \approx 6ET$ [5], and σ_{el} is the cross section for the $gg \to gg$ scattering (see Appendix B for details). The choice of the upper limit of integration is justified in the Appendix C. ϵ is then obtained as

$$\epsilon = \frac{C_A \alpha_s}{\pi^2} \times \langle q_\perp^2 \rangle \times \int_0^\pi d\theta \int_{-0.5}^{0.5} d\eta \left(\cosh\eta \int_{\Lambda\cosh\eta - \langle q_\perp \rangle \cos\theta}^{\frac{E}{\cosh\eta} - \langle q_\perp \rangle \cos\theta} \frac{d\omega}{\omega^2 + a^2} \right) =$$
$$= \frac{C_A \alpha_s}{\pi^2} \times \langle q_\perp^2 \rangle \times \int_0^\pi d\theta \left[\frac{1}{a} \int_{-0.5}^{0.5} d\eta \left(\cosh\eta \times \left\{ \tan^{-1} \frac{\frac{E}{\cosh\eta} - \langle q_\perp \rangle \cos\theta}{a} - \right.\right.\right. \tag{7}$$
$$\left.\left.\left. - \tan^{-1} \frac{\Lambda \cosh\eta - \langle q_\perp \rangle \cos\theta}{a} \right\} \right) \right].$$

For $\alpha_s = 0.3$ and $\Lambda = 0.3$ GeV, the energy loss per collision for a $E = 5$ GeV gluon is approximately 0.3 GeV.

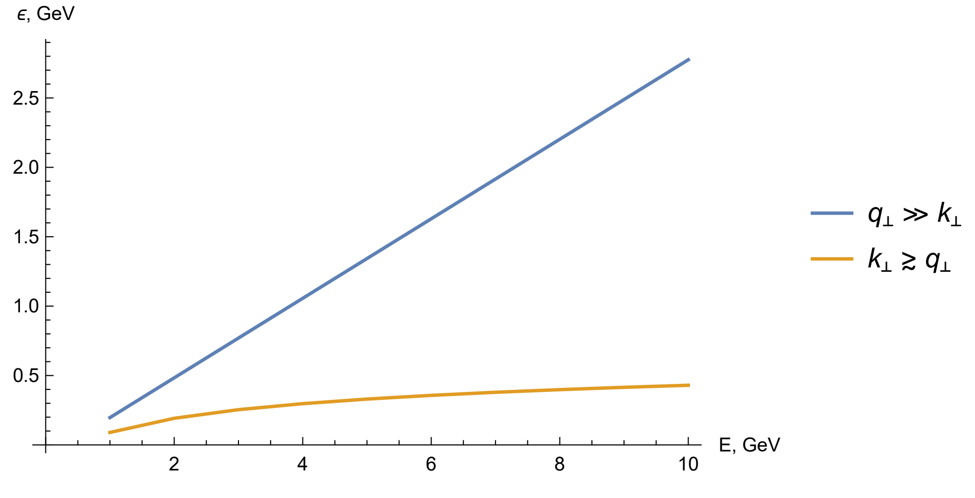

Figure 3: Energy loss per collision as a function of energy of the incoming high-energy gluon for the $gg \to ggg$ process ($\alpha_s = 0.3$, $\Lambda \approx T = 0.3$ GeV).

[5]In this report we consider $m_D \approx g(T)T$, where $g = \sqrt{4\pi\alpha_s}$ is the color charge. However, the Debye mass can be calculated from the quantum field theory [8].

As the energy loss of high-energy gluon is equal to the energy which is taken away by the emitted gluon, we can estimate the energy loss of high-energy gluon by multipling the interaction rate, Λ and the energy loss per collision, ϵ:

$$\boxed{-\frac{dE}{dx} = \epsilon \times \Lambda.} \tag{8}$$

For $\alpha_s = 0.3$ and $\Lambda = 0.3$ GeV, the energy loss per unit distance traveled for a $E = 5$ GeV gluon is approximately 0.4 GeV^2 for $q_\perp \gg k_\perp$ and 0.10 GeV^2 for $k_\perp \gtrsim q_\perp$.

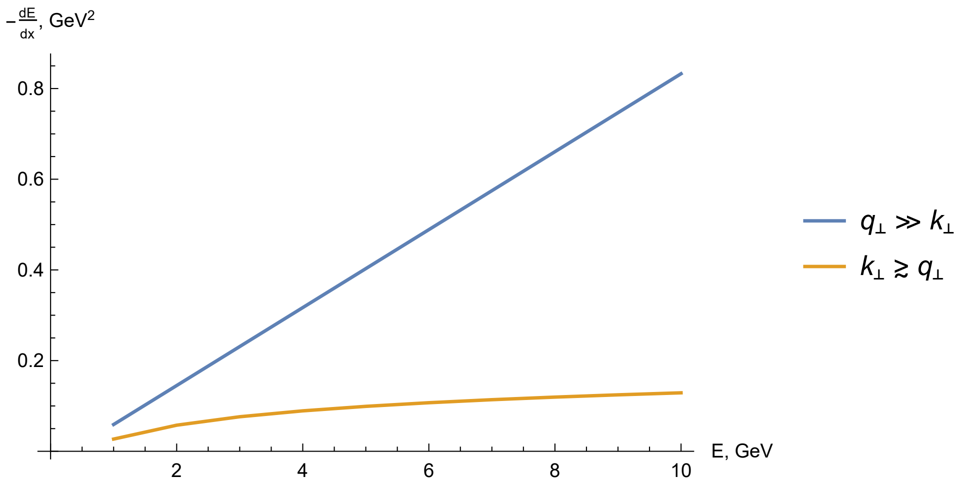

Figure 4: Energy loss per unit distance traveled as a function of energy of the incoming high-energy gluon for the $gg \to ggg$ process ($\alpha_s = 0.3$, $\Lambda \approx T = 0.3$ GeV).

In the limit of $q_\perp \gg k_\perp$, we observe a higher energy loss for gluons. This can be attributed to the dominance of soft gluons (characterized by small values of $k_\perp$) in the gluon spectrum.

2.2 Boltzmann transport equation and energy loss

In this paragraph, we follow the exposition presented in the work [9].

During the propagation through QGP heavy quarks (Q) dissipate energy by collisions with light quarks (q) and gluons (g): $Qq \to Qq$, $Q\bar{q} \to Q\bar{q}$, and $Qg \to Qg$.

The evolution of high-energy particles inside QGP is given by the Boltzmann transport equation:

$$\frac{df}{dt} = C[f], \tag{9}$$

where $f(\vec{x},\vec{p},t)$ is the one-particle phase-space distribution function of high-energy particles (in the present case f stands for heavy quarks distribution) and $C[f]$ is the collision term.

The assumption of uniformity in the plasma and absence of any external force leads to

$$\frac{\partial f}{\partial t} = C[f]. \tag{10}$$

The right-hand side of Eq. (10) represents a collision integral given by [10]

$$\frac{\partial f(\vec{p},t)}{\partial t} = \int d^3k \left[\underbrace{W(\vec{p}+\vec{k},\vec{k}) f(\vec{p}+\vec{k})}_{\text{Transition from } \vec{p}+\vec{k} \text{ to } \vec{p}\text{: ``gain''}} - \underbrace{W(\vec{p},\vec{k}) f(\vec{p})}_{\text{Transition from } \vec{p} \text{ to } \vec{p}-\vec{k}\text{: ``loss''}} \right]. \tag{11}$$

The function $W(\vec{p},\vec{k})$ is

$$W(\vec{p},\vec{k}) = \gamma \int \frac{d^3q}{(2\pi)^3} f'(\vec{q}) v \sigma_{(\vec{p},\vec{q})\to(\vec{p}-\vec{k},\vec{q}+\vec{k})}, \tag{12}$$

where $\vec{p}$ is the 3-momentum of the incoming high-energy particle, $\vec{k}$ is the 3-momentum transfer, $\vec{q}$ is the 3-momentum of the incoming particle of the medium, $f'(\vec{q})$ is the phase-space distribution of particles of the medium, in the present case it stands for light quarks and gluons (assumed to be position and time independent), v is the relative velocity, σ denotes the cross section, and γ is the spin and color degeneracy of particles of the medium.

Expanding the gain term about $\vec{p}$ to second order in $\vec{k}$ by considering small momentum transfer leads to the Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = \int d^3k \left[k_i \frac{\partial \left(W(\vec{p},\vec{k}) f(\vec{p},t) \right)}{\partial p_i} + \frac{1}{2} \sum_{i,j} k_i k_j \frac{\partial^2 \left(W(\vec{p},\vec{k}) f(\vec{p},t) \right)}{\partial p_i \partial p_j} \right] =$$
$$= \frac{\partial}{\partial p_i} \left(A_i(\vec{p}) f(\vec{p},t) \right) + \frac{\partial^2}{\partial p_i \partial p_j} \left(B_{ij}(\vec{p}) f(\vec{p},t) \right), \tag{13}$$

where we have introduced the transport coefficients of drag and diffusion, respectively:

$$A_i(\vec{p}) = \int d^3k k_i W(\vec{p},\vec{k}), \tag{14}$$
$$B_{ij}(\vec{p}) = \frac{1}{2} \int d^3k k_i k_j W(\vec{p},\vec{k}). \tag{15}$$

It is generally believed that the Fokker-Planck equation describes well the evolution of particles in a medium, which is in a state of thermal equilibrium.

The drag and diffusion coefficients are given by [10]

$$A_i(\vec{p}) \equiv \langle\!\langle p_i - p'_i \rangle\!\rangle, \tag{16}$$

$$B_{ij}(\vec{p}) \equiv \frac{1}{2}\langle\!\langle (p_i - p'_i)(p_j - p'_j) \rangle\!\rangle, \tag{17}$$

where $\vec{p}' = \vec{p} - \vec{k}$ is the 3-momentum of the scattered incoming high-energy particle.[6] This implies that the dot product of vectors p_i and A_i is

$$p_i A_i = \langle\!\langle p^2 - \vec{p}\cdot\vec{p}' \rangle\!\rangle. \tag{18}$$

The energy loss per unit distance traveled is equal to [11]

$$-\frac{dE}{dx} = \left\langle\!\!\left\langle \frac{E^2 - EE'}{p} \right\rangle\!\!\right\rangle, \tag{19}$$

where E and E' is the energy of the high-energy particle before and after the collision respectively.

So, a subtraction of Eq. (19) from Eq. (18) is

$$\begin{aligned} p_i A_i + p\frac{dE}{dx} &= \langle\!\langle p^2 - \vec{p}\cdot\vec{p}' - E^2 + EE' \rangle\!\rangle = -\langle\!\langle E(E - E') - \vec{p}\cdot(\vec{p} - \vec{p}') \rangle\!\rangle = \\ &= -\langle\!\langle p^\mu (p_\mu - p'_\mu) \rangle\!\rangle = -\frac{1}{2}\langle\!\langle (p_\mu - p'_\mu)^2 \rangle\!\rangle = -B_{00} + B_{ii}, \end{aligned} \tag{20}$$

where $B_{00} = \frac{1}{2}\langle\!\langle (E - E')^2 \rangle\!\rangle$ and $B_{ii} = \frac{1}{2}\langle\!\langle (p_i - p'_i)^2 \rangle\!\rangle$.

Thus,

$$-\frac{dE}{dx} = \frac{p_i A_i + B_{00} - B_{ii}}{p}. \tag{21}$$

Since A_i and B_{ij} depend only on the vector $\vec{p}$, they may be represented according to [10],

$$A_i(\vec{p}) = p_i A(p), \tag{22}$$

$$B_{ij}(\vec{p}) = \left(\delta_{ij} - \frac{p_i p_j}{p^2}\right) B_\perp(p) + \frac{p_i p_j}{p^2} B_\parallel(p). \tag{23}$$

Thus we can replace $p_i A_i$ in Eq. (21) with

$$p_i A_i = p^2 A(p). \tag{24}$$

[6] $\langle\!\langle p_i - p'_i \rangle\!\rangle = \frac{1}{2E}\int \frac{d^3q}{(2\pi)^3 2E_q}\int \frac{d^3q'}{(2\pi)^3 2E_{q'}}\int \frac{d^3p'}{(2\pi)^3 2E'} \times \frac{1}{g}\sum |\mathcal{M}|^2 (2\pi)^4 \delta^4(p + q - p' - q')(p_i - p'_i) f'(\vec{q})(1 + f'(\vec{q}'))$, where g is the spin and color degeneracy of the high-energy particles, $\mathcal{M}$ is the matrix element of a two-body collision and the terms of the sum range over the initial and final spin and color states.

So finally,

$$\boxed{-\frac{dE}{dx} = \frac{p^2 A + B_{00} - B_{ii}}{p}.} \tag{25}$$

We consider now the drag and diffusion coefficients due to collisional processes for heavy quarks interacting with the QGP at temperature $T = 0.35$ GeV calculated following [10].

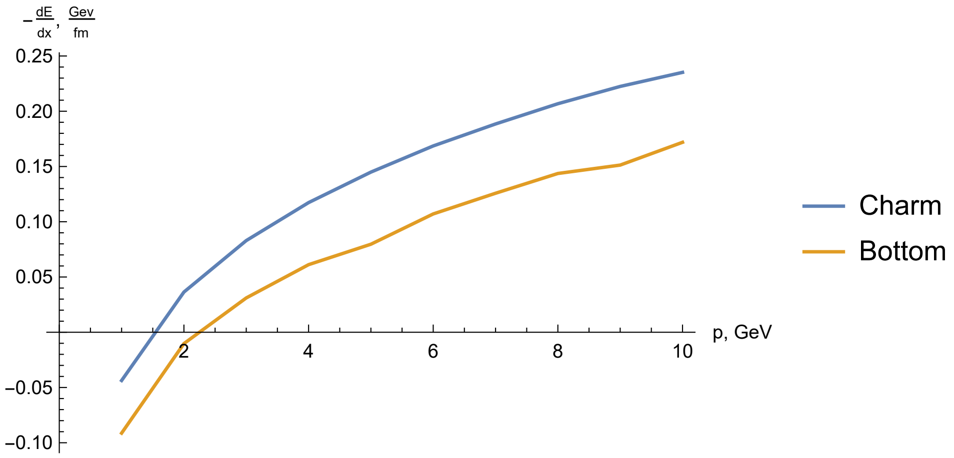

Figure 5: Energy loss per unit distance traveled as a function of momentum of the incoming charm quark $m_c = 1.5$ GeV (blue) or bottom quark $m_b = 4.2$ GeV (yellow) for the $2 \to 2$ processes in the QGP at temperature $T = 0.35$ GeV.

For $T = 0.35$ GeV, the energy loss per unit distance traveled for a $p = 5$ GeV charm quark is approximately $0.14\ \frac{\text{GeV}}{\text{fm}}$ and $0.08\ \frac{\text{GeV}}{\text{fm}}$ for a $p = 5$ GeV bottom quark. So, *quarks that have a larger mass lose less energy.*

Summary

In this project, numerical estimates of the radiative energy loss of gluons inside a gluonic plasma are made:

- For $\alpha_s = 0.3$ and $\Lambda \approx T = 0.3$ GeV, the energy loss per collision for a $E = 5$ GeV gluon is approximately 1.3 GeV for $q_\perp \gg k_\perp$ and 0.3 GeV for $k_\perp \gtrsim q_\perp$. We have shown that *increasing the strong coupling (α_s), the interaction time (Λ^{-1}) or the energy of the incoming high-energy gluon (E) results in an increase the energy loss per collision*;

- For $\alpha_s = 0.3$ and $\Lambda \approx T = 0.3$ GeV, the energy loss per unit distance traveled for a $E = 5$ GeV gluon is approximately 0.4 GeV^2 for $q_\perp \gg k_\perp$ and 0.10 GeV^2 for $k_\perp \gtrsim q_\perp$.

In the limit of $q_\perp \gg k_\perp$, we observe a higher energy loss for gluons. This can be attributed to the dominance of soft gluons (characterized by small values of $k_\perp$) in the gluon spectrum.

Energy loss of heavy quarks (charm and bottom) interacting with light quarks and gluons are performed:

- For $T = 0.35$ GeV, the energy loss per unit distance traveled for a $p = 5$ GeV charm quark is approximately 0.14 $\frac{\text{GeV}}{\text{fm}}$ and 0.08 $\frac{\text{GeV}}{\text{fm}}$ for a $p = 5$ GeV bottom quark. So, *quarks that have a larger mass lose less energy.*

Using the approach and results of the energy loss calculation in this study, one can probe the properties of the QGP medium, such as its energy density [12], temperature [13], and viscosity [14]. It also provides evidence for the existence of the QGP itself, as the observed energy loss and modifications of the jet properties (known as jet quenching [15]) can be attributed to the interactions with the QGP medium.

A Appendix

We present a conclusion on the formation time in QED that is similar to [5].

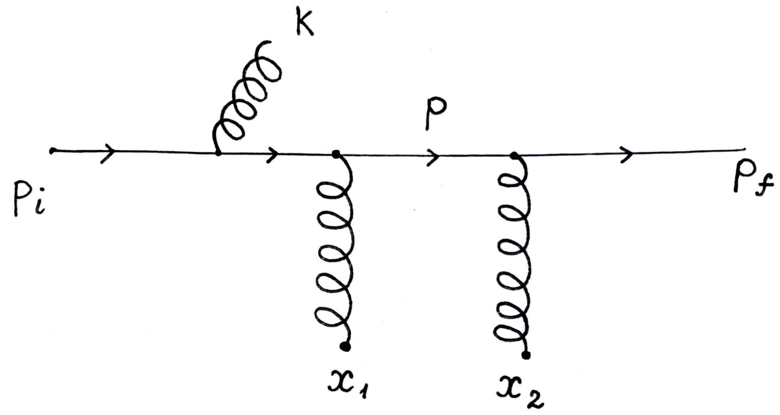

Figure 6: One of the Feynman diagrams for gluon radiation from the parton line induced by double scattering at static centers.

The radiation amplitude associated with double scattering is

$$\mathcal{R}_2^{\text{QED}} = ig\left[\left(\frac{\varepsilon\cdot p_i}{k\cdot p_i} - \frac{\varepsilon\cdot p}{k\cdot p}\right)e^{ik\cdot x_1} + \left(\frac{\varepsilon\cdot p}{k\cdot p} - \frac{\varepsilon\cdot p_f}{k\cdot p_f}\right)e^{ik\cdot x_2}\right], \tag{26}$$

where $k = (\omega, \vec{k})$ is the 4-momentum of the emitted gluon, ε is the 4-polarization of the emitted gluon, p_i is the initial 4-momentum of the high-energy parton, p_f is the final 4-momentum of the high-energy parton, p is the 4-momentum of the intermediate parton, $x_1 = (0, \vec{x_1})$ and $x_2 = (t_2, \vec{x_2})$ are the 4-coordinates of two potentials with $t_2 = \frac{z_2 - z_1}{v_z} = \frac{L}{v_z}$, and v_z is the longitudinal velocity of the high-energy parton. We consider the scattering centers to be distributed along the z-axis.

The interference between two scatterings is determined by the relative phase factor:

$$k\cdot(x_2 - x_1) = \omega t_2 - \vec{k}(\vec{x_2} - \vec{x_1}) = L\left(\frac{\omega}{v_z} - k_z\right) \tag{27}$$

with the formation time $\tau_{\text{QED}} = \frac{1}{\frac{\omega}{v_z} - k_z} = \frac{1}{\frac{\omega}{v_z} - \sqrt{\omega^2 - k_\perp^2}}$. For $v_z \to 1$ and $k_\perp \ll \omega$, $\tau_{\text{QED}} \approx \frac{2\omega}{k_\perp^2}$.

B Appendix

For dominant small-angle $gg \to gg$ scattering [5],

$$\frac{d\sigma_{el}}{dq_\perp^2} = \frac{9\pi\alpha_s^2}{2q_\perp^4}. \tag{28}$$

σ_{el}, the cross section for the $gg \to gg$ scattering is then obtained as

$$\sigma_{el} = \int_{m_D^2}^{\frac{s}{4}} dq_\perp^2 \frac{9\pi\alpha_s^2}{2q_\perp^4} = \frac{9\pi\alpha_s^2}{2} \frac{s - 4m_D^2}{sm_D^2} \tag{29}$$

and $\langle q_\perp \rangle$ is

$$\langle q_\perp \rangle = \frac{1}{\sigma_{el}} \int_{m_D^2}^{\frac{s}{4}} dq_\perp^2 \frac{9\pi\alpha_s^2}{2q_\perp^3} = \frac{9\pi\alpha_s^2}{\sigma_{el}} \frac{\sqrt{s} - 2m_D}{\sqrt{s}m_D} = \frac{2\sqrt{s}m_D}{\sqrt{s} + 2m_D}. \tag{30}$$

Similarly we can get

$$\langle q_\perp^2 \rangle = \frac{sm_D^2}{s - 4m_D^2} \ln\left(\frac{s}{4m_D^2}\right). \tag{31}$$

C Appendix

Consider the reaction $g(k_1) + g(k_2) \to g(k_3) + g(k_4) + g(k_5)$, where 4-momentum of gluons in the center-of-momentum frame are defined by

$$k_1 = (E, \vec{0}, k_{1z}), \tag{32}$$
$$k_2 = (E_2, \vec{0}, -k_{1z}), \tag{33}$$
$$k_3 = (E_3, \vec{q}_\perp, k_{3z}), \tag{34}$$
$$k_4 = (E_4, -\vec{q}_\perp, -k_{3z}), \tag{35}$$
$$k_5 = (k_0, \vec{k}_\perp, k_z), \tag{36}$$

where $|\vec{k_5}| \ll |\vec{k_3}|, |\vec{k_4}|$ in the soft gluon radiation limit. We may also approximately assume $E \approx E_2$, $E_3 \approx E_4$ for high-energy regime.

The Mandelstam variable s' is given by

$$s' = (k_3 + k_4)^2 = (E_3 + E_4)^2 \approx 4E_3^2. \tag{37}$$

From the other hand it is

$$s' = 2(k_3 \cdot k_4) = 2(E_3 E_4 + q_\perp^2 + k_{3z}^2) \approx 2(E_3^2 + q_\perp^2 + k_{3z}^2). \tag{38}$$

Note that the Mandelstam variables for the soft gluon emission obey the constraint equation:

$$s + t + u + s' + t' + u' = 0. \tag{39}$$

Thus,

$$\begin{aligned} s' = -s - t - u - t' - u' = 2(k_1 \cdot k_5) + s - 2(k_2 \cdot k_5) = \\ = 2(Ek_0 - k_{1z}k_{5z}) + s - 2(E_2 k_0 + k_{1z}k_{5z}) \approx s. \end{aligned} \tag{40}$$

So finally,

$$q_\perp^2 \approx \frac{s}{2} - E_3^2 - k_{3z}^2 \approx \frac{s}{2} - \frac{s}{4} - k_{3z}^2 = \frac{s}{4} - k_{3z}^2 \leqslant \frac{s}{4}. \tag{41}$$

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