JOINT INSTITUTE FOR NUCLEAR RESEARCH
Veksler and Baldin laboratory of High Energy Physics

FINAL REPORT ON THE INTEREST PROGRAMME

Numerical Methods in Theory of Topological Solitons

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Participation period:
October 30 – December 10, Wave 9

Dubna, 2023

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Abstract

Soliton solutions of O(3) Non-linear Sigma Model and Baby Skyrme Model in (2+1) dimensions were studied. For O(3) non-linear sigma model, north/south pole stereographic projections were used to construct multi-soliton solutions of topological charges, Q =2, 8, and generated plots for their energy densities. For baby skyrme model, its numerical solution of the profile function, f(r) of the hedgehog ansatz is solved numerically as a boundary value problem for the following rescaled mass parameters, μ = 0.01, 0.1, and 1.

1 Introduction

Solitons arose from solitary waves / waves of translation which were discovered by John Scott Russell at Union Hermiston (Near Edinburgh) in 1834 while conducting experiments for canal boats [1]. Soliton is a special solution for special non-linear Partial Differential Equations (PDEs), field equations, and different types of models which are used to describe the elementary particles in particle physics for example. Soliton solutions are localized (A lump of energy), non-dispersive, meaning that it propagates through with constant shape, and they are preserved with collisions with other solitons. In field theories, solitons are constructed by field potentials with different Lagrangian densities. Different field potential yields different soliton solutions. Solitons have many applications including but not limited to particle physics, biological physics, non-linear optics, algebraic geometry, and group theory.

2 O(3) Non-linear Sigma Model

A sigma model is a scale invariant model that is used to describe the strong interaction between pions and nucleons. In this model, soliton solutions can be altered by scale transformations, thus, these solutions do not obey the properties of a soliton, these solutions are named 'lumps' instead. In this model, a real scalar field is a map from a N-dimensional Riemannian manifold to a M-dimensional target space [2]. In this section, (2+1) dimensional O(3) sigma model is studied with mapping from a unit sphere, S^2 onto the plane R^2.

The Lagrangian is of the form

$$L = \frac{1}{4}\left(\partial_\mu \phi^a\right)^2 \quad , \tag{1}$$

with the constraint

$$\phi^a \cdot \phi^a = 1 \quad , \tag{2}$$

on the field triplet, $\phi^a = (\phi^1, \phi^2, \phi^3)$.

2.1 North & South Pole Stereographic Projections

The constraint given in Equation (2) determines a unit sphere on a 3-dimensional field space, this sphere is positioned at the origin, and it can be described by using 2 independent variables, thus, the field space dimension can be decreased to fit the sphere's dimension. The mapping from the sphere to the plane, can be found by using the equation of a straight line in 3 dimensions, except the points at the north and south poles.

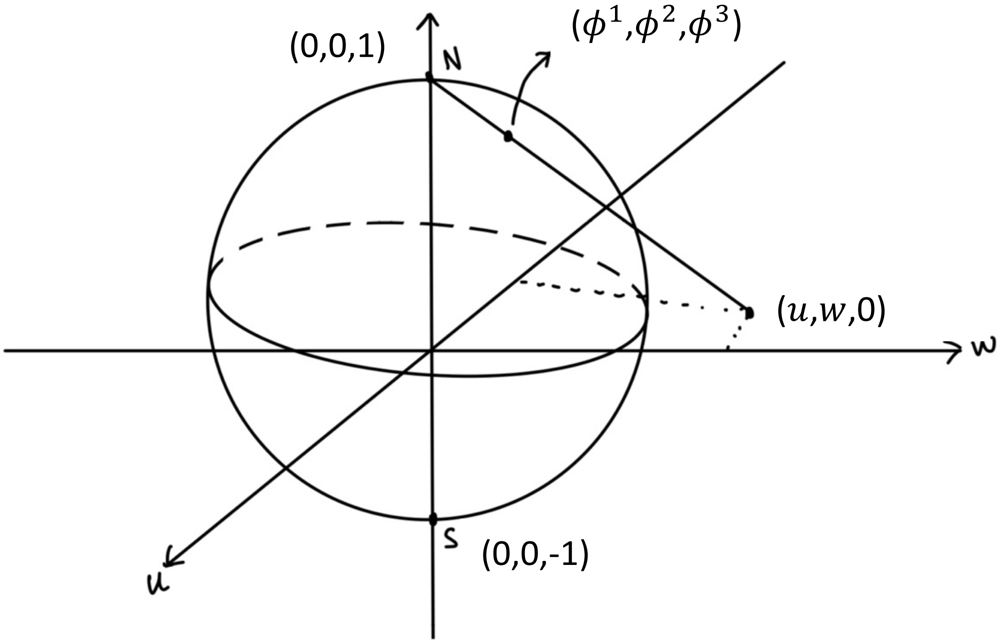

Figure 1. North Pole Stereographic Projection

The equation of a straight line in 3D is of the form

$$\begin{pmatrix} u \\ w \\ 0 \end{pmatrix} = \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix} + \lambda \begin{pmatrix} u \\ w \\ -1 \end{pmatrix}. \tag{3}$$

Thus, one can find the mapping from the North pole given by

$$(u, w) = \left(\frac{\phi_1}{1+\phi_3}, \frac{\phi_2}{1+\phi_3} \right). \tag{4}$$

A complex variable W, can be defined by

$$\mathrm{W} = \frac{\phi_1 + \mathrm{i}\phi_2}{\phi_3}. \tag{5}$$

Inverse transformations can be obtained by using the constraint from Equation (2), by substituting in the field components, one can obtain the following relations

$$\phi^1 = \frac{2u}{1+u^2+w^2}\,, \tag{6}$$

$$\phi^2 = \frac{2w}{1+u^2+w^2}\,, \tag{7}$$

$$\phi^3 = \frac{u^2+w^2-1}{1+u^2+w^2}\,. \tag{8}$$

Define W' to be the complex variable projected from the South Pole as

$$\mathrm{W}' = \frac{\phi_1'+\mathrm{i}\phi_2'}{\phi_3'}\,. \tag{9}$$

Thus, the inverse transformations are of the form

$$\phi^{1\,\prime} = \frac{2u\prime}{1+u^{2\,\prime}+w^{2\,\prime}} \tag{10}$$

$$\phi^{2\,\prime} = \frac{2w\prime}{1+u^{2\,\prime}+w^{2\,\prime}} \tag{11}$$

$$\phi^{3\,\prime} = -\frac{u^{2'}+w^{2'}-1}{1+u^{2\,\prime}+w^{2\,\prime}} \tag{12}$$

As the field potentials do not depend on the type of projection, meaning that

$$\begin{gathered}\phi^1 = \phi^{1'},\\ \phi^2 = \phi^{2'},\\ \phi^3 = \phi^{3'},\end{gathered}$$

one can then discover the connection between these two projections is given by

$$W_S = \frac{1}{\overline{W_N}}\,. \tag{13}$$

The Lagrangian can be rewritten in terms of the complex variable, W, as

$$L = \frac{\partial_\mu W \partial^\mu \overline{W}}{1+\left(W\overline{W}\right)^2}\,. \tag{14}$$

This form of Lagrangian in terms of the complex variable z = x+iy and $\overline{z}$ = x-iy is given by

$$L = \frac{|\partial_z W|^2 + |\partial_{\overline{z}} W|^2}{(1+|W|^2)^2}, \tag{15}$$

where $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, and $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$.

By using the Euler-Lagrange Equation, one can find the following field equation [2]

$$W_{z\overline{z}} = 2\overline{W}\frac{W_z W_{\overline{z}}}{(1+W\overline{W})^2}, \tag{16}$$

Soliton solutions are the absolute minimum of the total energy and are produced via a holomorphic map W(z) for Q > 0 and $W(\overline{z})$ for Q < 0.

W(z) is defined by

$$W(z) = \lambda\frac{P(z)}{Q(z)}, \tag{17}$$

where P(z) and W(z) are polynomials of degree at most N and at least one of them is at degree N and lambda is an arbitrary complex number.

2.2 Soliton Solutions

2.2.1 Two Soliton Solutions

The following holomorphic map [2] is considered to construct a 2 soliton solution

$$W(z) = \frac{(z-a)(z-b)}{(z-c)(z-d)} \tag{18}$$

where a, b, c, and d are arbitrary complex numbers, and they define the positions of the solitons.

Different values of a, b, c, and d are examined to observe different types of two soliton solutions. Plots are generated using Mathematica.

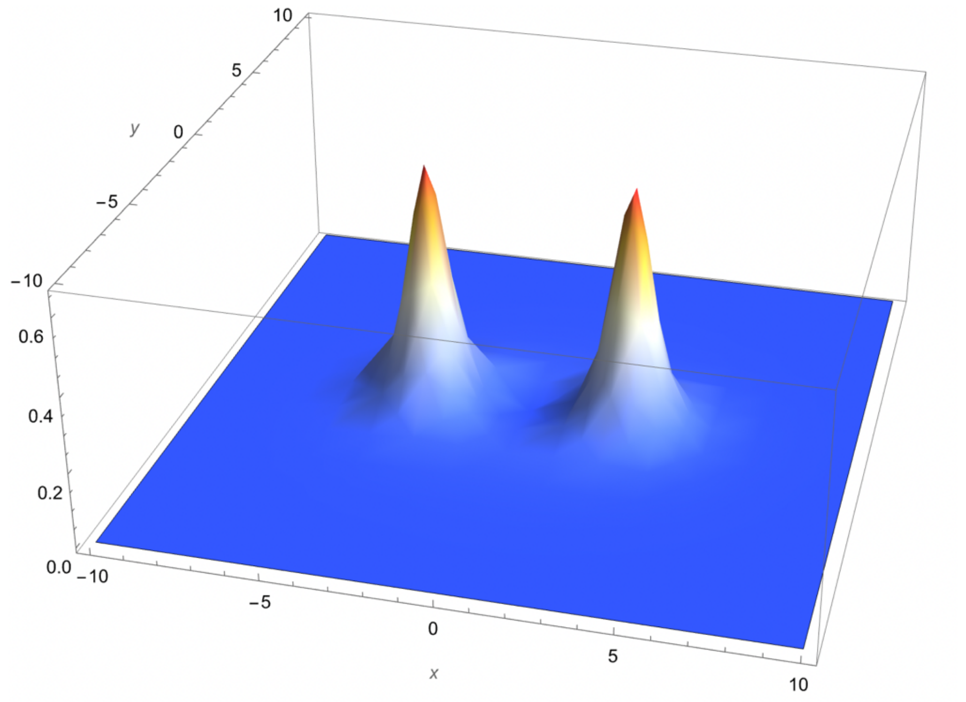

Figure 2. Q=2 Energy Density Distribution with a=1, b=4, c=-1, d=-4

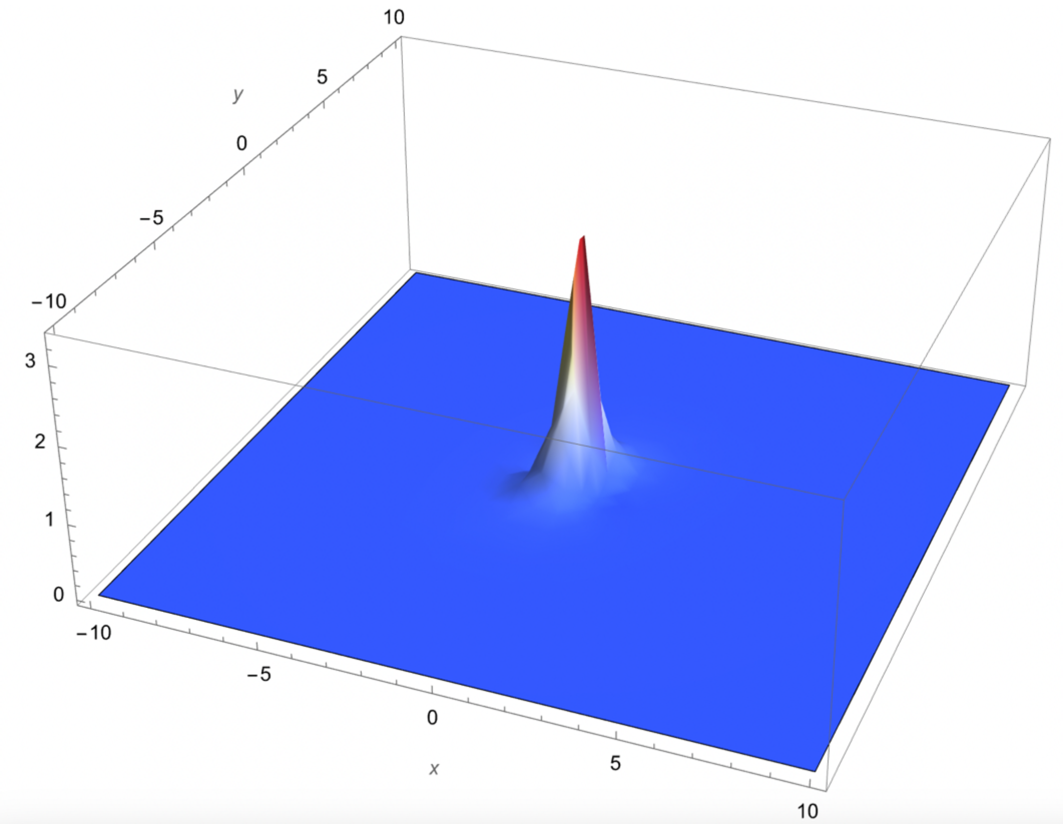

Figure 3. Q=2 Energy Density Distribution with a=1+I, b=-2.5, c=-I, d=-0.25

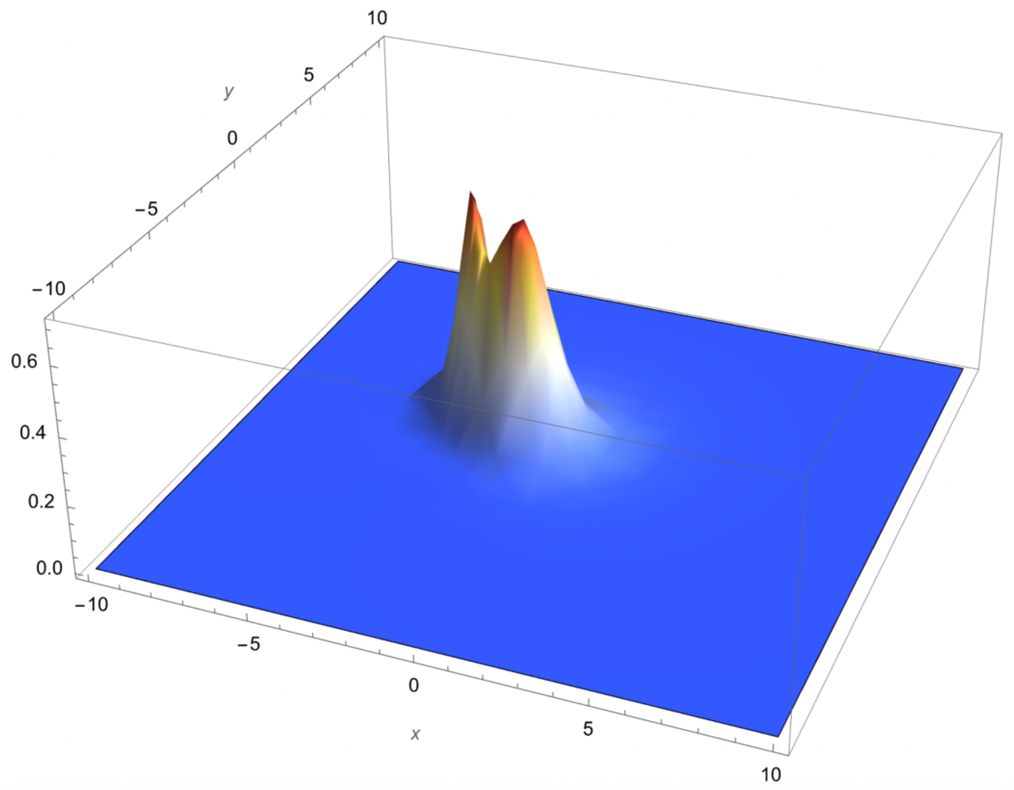

Figure 4. Q=2 Energy Density Distribution with a=i-3.5, b=-1.75, c=0.45, d=3.5i-2.3

2.1.1 Eight Soliton Solutions

The following holomorphic map is considered to construct an 8 soliton solution

$$W_N = \frac{4}{\frac{1}{z}+\frac{1}{z+\frac{1}{2}-i}+\frac{1}{z-\frac{1}{2}-i}+\frac{1}{z+1}+\frac{1}{z-1}+\frac{1}{z+\frac{3}{2}+i}+\frac{1}{z-\frac{3}{2}+i}+\frac{1}{z-2i}} \, , \tag{19}$$

and the equivalent south pole map is given by

$$W_S = \frac{1}{4}\left(\frac{1}{\bar{z}} + \frac{1}{\bar{z}+\frac{1}{2}+i} + \frac{1}{\bar{z}-\frac{1}{2}+i} + \frac{1}{\bar{z}+1} + \frac{1}{\bar{z}-1} + \frac{1}{\bar{z}+\frac{3}{2}-i} + \frac{1}{\bar{z}-\frac{3}{2}-i} + \frac{1}{\bar{z}+2i}\right) , \tag{20}$$

via Equation (13).

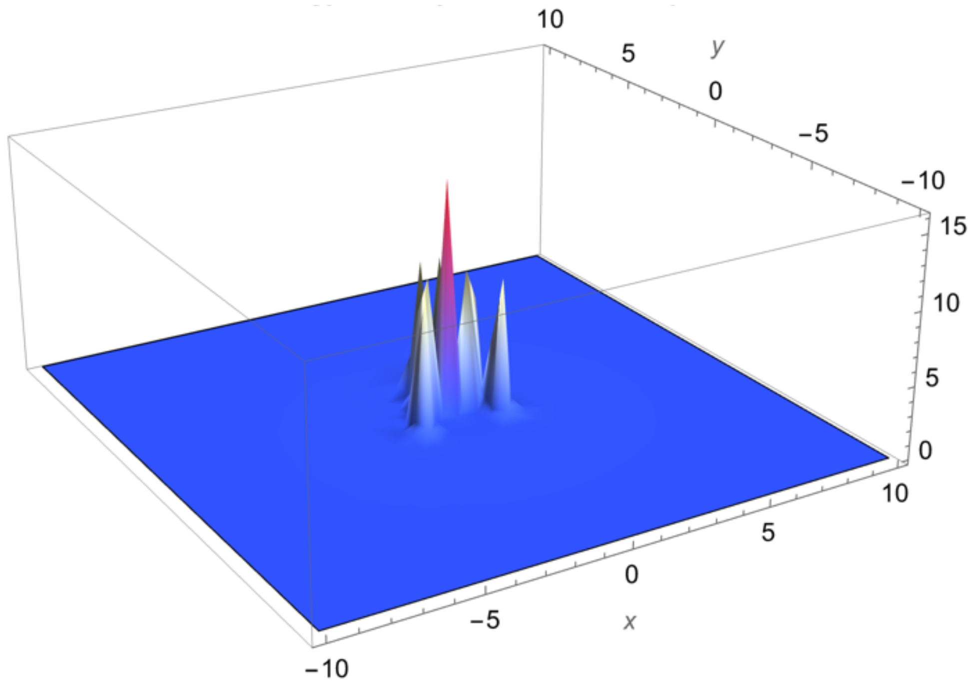

Figure 5. Q=8 Energy Density Distribution - Pyramid

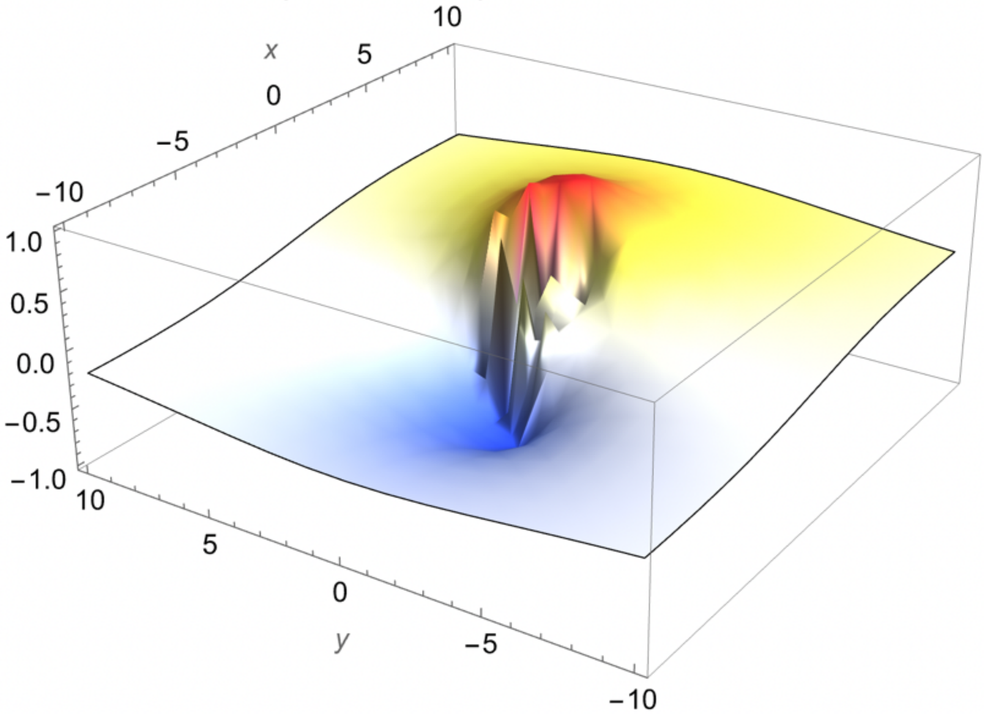

Figure 6. ϕ^1 Field Component Distribution

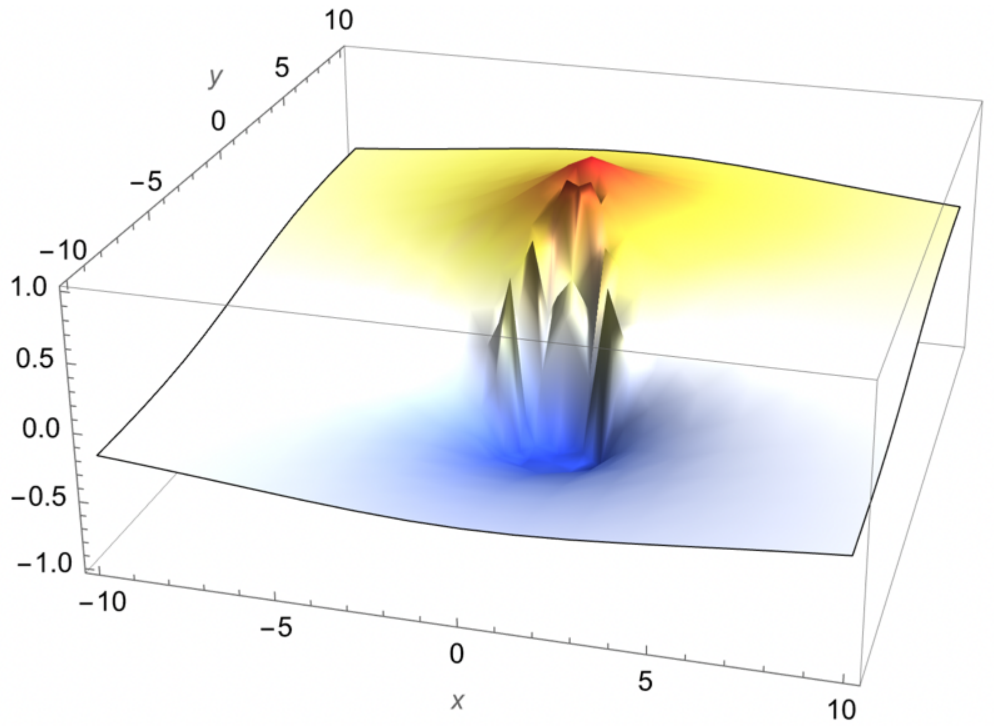

Figure 7. ϕ^2 Field Component Distribution

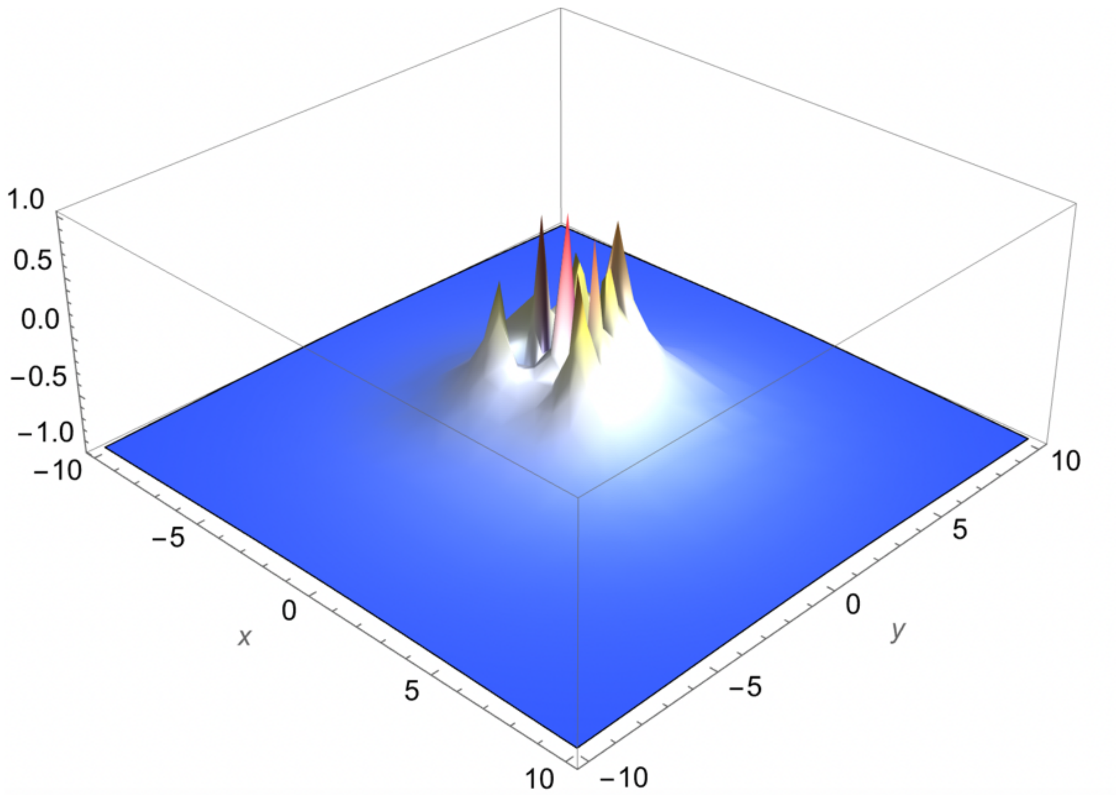

Figure 8. ϕ^3 Field Component Distribution

A chain of 8 solitons aligned on the x-axis is given by the mapping of the form

$$W_N = \frac{1}{\sum_{d=0}^{4} \frac{1}{z+d} + \sum_{d=-3}^{-1} \frac{1}{z+d}} \, . \tag{21}$$

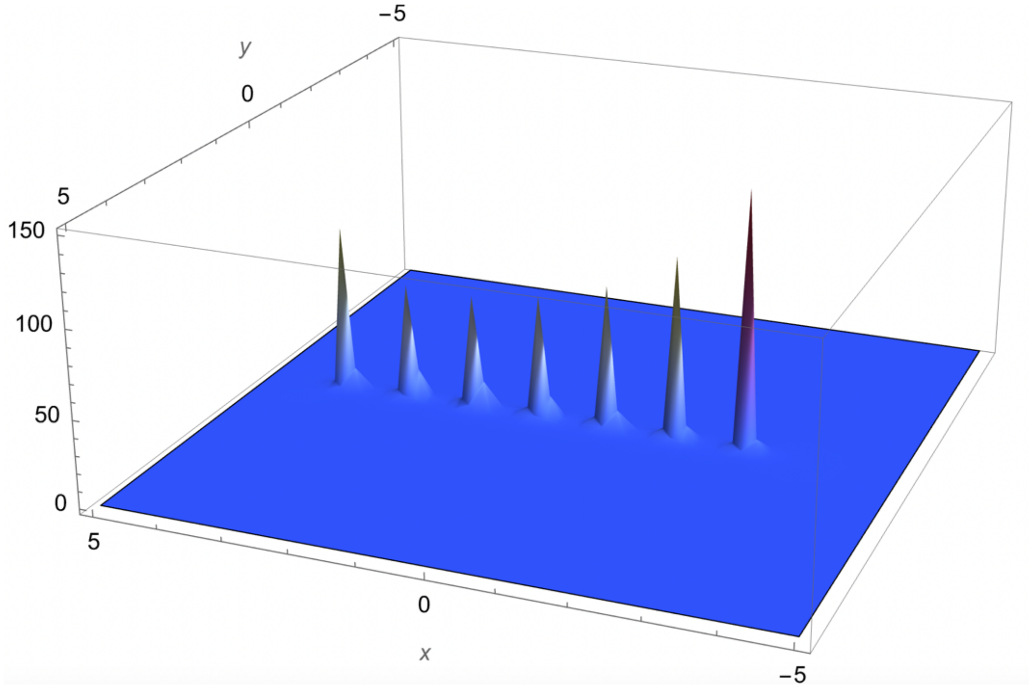

Figure 9. Q=8 Energy Distribution - Line

3 Baby Skyrme Model

A Skyme Model is a nucleonic model that is used to describe baryons as topological solitons, known as skyrmions [2]. Skyme models have many applications including but not limited to condensed matter physics, and gravitational physics. In this section, we study the (2+1) dimensional baby skyrme model with the Lagrangian density of

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi^a\right)^2 - \frac{1}{4}\left(\epsilon_{abc}\phi^a \partial_\mu \phi^b \partial_\nu \phi^c\right)^2 - \mu^2(1-\phi_3), \tag{22}$$

where $\phi^a = (\phi^1, \phi^2, \phi^3)$ are the field triplet with the constraint from Equation (2).

The field is a topological map $\phi: S^2 \to S^2$ and the topological charge is given by

$$Q = \int \frac{1}{4}\left(\epsilon_{abc}\phi^a \partial_\mu \phi^b \partial_\nu \phi^c\right) dS. \tag{23}$$

The energy functional is given by

$$E = \int \mathcal{L}\, d^2x\ . \tag{24}$$

For soliton solutions in this model, $E\ \neq 4\pi|Q|$, however, isorotational O(2) symmetry takes place when Q = 1, thus, the following hedgehog ansatz can be considered

$$\vec{\phi} = (cos\theta sinf(r), sin\theta f(r), cosf(r)), \tag{25}$$

where r and θ are in plane polar coordinates and f(r) is the profile function and is a monotonically decreasing function [2].

By substituting Equation (27) into Equation (26), one can obtain the following parametrization of energy functional

$$E = 2\pi \int_0^\infty dr\ (\tfrac{r}{2} f'(r)^2 + \frac{\sin^2 f(r)}{2r}(1 + f'(r)^2) + r\mu^2(1 - cosf(r))). \tag{26}$$

By using Euler-Lagrange Equation, one can find the following

$$\left(r + \frac{\sin^2 f(r)}{r}\right) f''(r) + \left(1 - \frac{\sin^2 f(r)}{r^2} + \frac{f'(r)sinf(r)cosf(r)}{r}\right) f'(r) - \frac{sinf(r)cosf(r)}{r} - r\mu^2 sinf(r) = 0. \tag{27}$$

This equation's solution can be considered as a boundary value problem, one can solve it numerically using a shooting method with 4th Runge Kutta Integration in python. The boundary conditions are given by $f(\infty) = 0, and\ f(0) = \pi.$

To evaluate the value of the topological charge, one can perform the following integration, using the hedgehog ansatz.

$$Q = \int \frac{1}{4}(\epsilon_{abc}\phi^a \partial_\mu \phi^b \partial_\nu \phi^c)\, dS$$

$$= \frac{2}{8\pi} \iint_{0\ 0}^{\infty\ 2\pi} drd\theta\ f'(r)(\sin^3 f(r) + \cos^2 f(r) sinf(r))$$

$$= \frac{1}{2}\int_0^\infty dr\ sinf(r) f'(r)$$

$$= 1$$

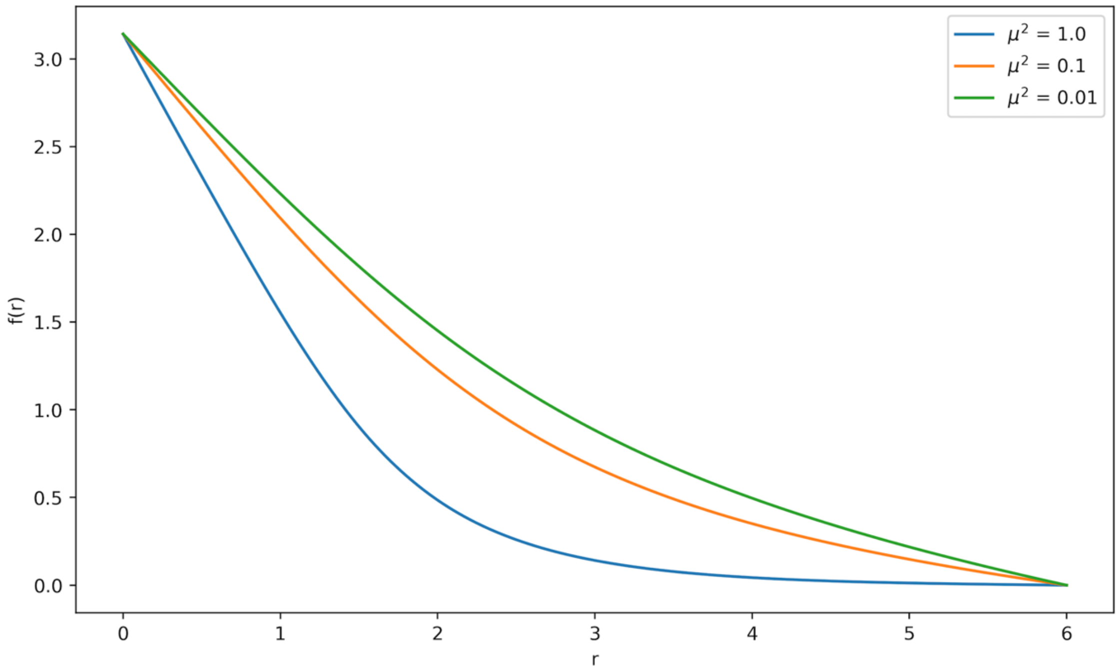

Figure 10. f(r) profile function plot with re-scaled mass parameters, $\mu^2 = 1.0, 0.1, 0.01$.

4 Conclusion

For the (2+1) dim O(3) sigma model, its energy density and potential plots of 2 soliton and 8 soliton solutions were successfully created using north/south poles stereographic projections. The field potentials do not depend on the type of projection. One can use a similar mapping for the 2 and 8 soliton solutions to generalise the solution to N number of solitons. For the (2+1) dim baby skyrme model, f(r) is successfully solved numerically in python, and the plot is as expected. As the re-scaled mass parameter increases, f(r) decreases faster.

References

[1] *The early history of Solitons (solitary waves) - iopscience*. Available at: https://iopscience.iop.org/article/10.1088/0031-8949/57/3/016

[2] Y. M. Shnir. Topological and non-topological solitons in scalar field theories. Cambridge University Press, 2018.