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“Multiplicity processes”

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Content

1. Introduction 3
2. Quark and gluon jets 3
3. Explicit solutions in particular cases 9
4. Sources 11

1. Introduction

Nowadays particle physics became incredibly popular. There are more and more accelerators and projects with higher and higher energies. With energy increasing new channels of decays were discovered, new particles were born. It has become stimulus for development of new theories and models. In particular, the quantum chromodynamics (QCD) was created. Studying of strong interaction can provide deep understanding of matter and energy.

The main problem of high energy physics is large number of secondary particles. Multiparticle production contains a lot of information about nature of strong interaction. Analysis of MP process is carried out using statistical methods. During investigation of MP processes jets were discovered. Jets phenomena can be studied in all processes where energetic partons (quarks and gluons) are produced. However, the most common is e^+e^- annihilation at high energy.

$$e^+e^- \to \gamma/Z^0 \to \bar{q}q$$

The first stage of fission partons at high energy is called *a stage of cascade*. When energy of partons decreases they must form the hadrons which are observable. This is the second stage – *a stage of hadronization*. It cannot be described with perturbative QCD while the first stage can.

Multiparton spectra inside QCD gluon and quark jets have been studied by Konishi, Ukawa and Veneziano. The results support the idea that gluon jets are softer than quark jets and there is order among final partons.

2. Quark and gluon jets

To study multiparticle production we used approach of A. Giovannini [1]. The main idea is to describe quark and gluon jets and their development through subnuclear matter as Markov branching processes.

It was proposed to interpret the natural QCD evolution parameter

$$Y = \frac{1}{2\pi b} log\left[1 + \alpha b\, log\frac{Q^2}{\mu^2}\right] \tag{2.1}$$

as the thickness of the QCD jets. Here $2\pi b = \frac{1}{6}\left(11N_c - 2N_f\right)$ for a theory with N_c colors and N_f flavours.

There are 3 main elementary processes in QCD jets that contribute with different weight:

1) A: gluon fission (g → g + g)
2) Ã: quark bremsstrahlung (q → q + g)
3) B: quark pair creation (g → q + q̅)

The probability for a gluon or quark to convert into m_q quarks and m_g gluons in the (Y, $Y+\Delta Y$) can be given by sum of probabilities:

$$\text{(gluon)} \qquad \delta_{1,m_g}\delta_{0,m_q} + a^{(g)}_{m_g,m_q}\Delta Y + o(\Delta Y) \tag{2.2}$$

$$\text{(quark)} \qquad \delta_{0,m_g}\delta_{1,m_q} + a^{(q)}_{m_g,m_q}\Delta Y + o(\Delta Y) \tag{2.3}$$

Due to only 3 processes are being allowed in the same interval ΔY, we get for gluon (4) and in case of quark (5)

$$1 + a^{(g)}_{1,0}\Delta Y + a^{(g)}_{2,0}\Delta Y + a^{(g)}_{0,2}\Delta Y + o(\Delta Y) \tag{2.4}$$

$$1 + a^{(q)}_{0,1}\Delta Y + a^{(q)}_{1,1}\Delta Y + o(\Delta Y) \tag{2.5}$$

Notice that $a^{(g)}_{1,0} + a^{(g)}_{2,0} + a^{(g)}_{0,2} = 0$ and $a^{(q)}_{0,1} + a^{(q)}_{1,1} = 0$ because of probability conservation. Let's identify $a^{(g)}_{2,0}$ as A, $a^{(g)}_{0,2}$ as Ã, and $a^{(q)}_{1,1}$ as B. Now we get AΔY as the probability that a gluon in the infinitesimal interval ΔY will convert into 2 gluons; ÃΔY as the probability that a quark will radiate a gluon; and BΔY as the probability that a quark-antiquark pair will be born from gluon. We assumed that A, Ã and B are Y-independent constants, and each parton act independently from the others, always with the same probability. After this, the infinitesimal function for gluon (6) and quark (7) jets are introduced:

$$w^{(g)}(u_g, u_q) = \sum_{m_g,m_q=0}^{\infty} a^{(g)}_{m_g,m_q} u_g^{m_g} u_q^{m_q} = a^{(g)}_{2,0}u_g^2 + a^{(g)}_{0,2}u_q^2 + a^{(g)}_{1,0}u_g$$

$$w^{(g)}(u_g, u_q) = Au_g^2 + Bu_q^2 - (A+B)u_g \tag{2.6}$$

$$w^{(q)}(u_g, u_q) = \sum_{m_g,m_q=0}^{\infty} a^{(q)}_{m_g,m_q} u_g^{m_g} u_q^{m_q} = a^{(q)}_{1,0}u_q + a^{(q)}_{1,1}u_g u_q$$

$$w^{(q)}(u_g, u_q) = \tilde{A}(u_g u_q - u_q) \tag{2.7}$$

Giovannini defines $P_{m_g,m_q;n_g,n_q}(Y)$ as the probability that m_g gluons and m_q quarks will be transformed into n_g gluons and n_q quarks over a jet of thickness Y. Probability generating function for a gluon jet (8) and a quark jet (9) will be

$$G(u_g, u_q; Y) = \sum_{n_g, n_q = 0}^{\infty} P_{1,0;n_g,n_q}(Y) u_g^{n_g} u_q^{n_q} \tag{2.8}$$

$$Q(u_g, u_q; Y) = \sum_{n_g, n_q = 0}^{\infty} P_{0,1;n_g,n_q}(Y) u_g^{n_g} u_q^{n_q} \tag{2.9}$$

Action of different partons are independent: from a probabilistic point of view the total m_g gluons and m_q quarks populations are evolving as $(m_g + m_q)$ independent parton populations, each with one initial quark or gluon. This fact summarizes the branching Markov chain nature of the process. It can be shown through straightforward calculations that

$$\sum_{n_g, n_q = 0}^{\infty} P_{m_g,m_q;n_g,n_q}(Y) u_g^{n_g} u_q^{n_q} = \left[G(u_g, u_q; Y)\right]^{m_g} \left[Q(u_g, u_q; Y)\right]^{m_q} \tag{2.10}$$

Moreover, since the process is homogenous in Y the transition probabilities obey Chapman-Kolmogorov equations in general case

$$P_{m_g,m_q;n_g,n_q}(Y + Y') = \sum_{l_g, l_q = 0}^{\infty} P_{m_g,m_q;l_g,l_q}(Y) P_{l_g,l_q;n_g,n_q}(Y') \tag{2.11}$$

and in case of gluon jet and quark jet respectively

$$P_{1,0;n_g,n_q}(Y + Y') = \sum_{l_g, l_q = 0}^{\infty} P_{1,0;l_g,l_q}(Y) P_{l_g,l_q;n_g,n_q}(Y') \tag{2.12}$$

$$P_{0,1;n_g,n_q}(Y + Y') = \sum_{l_g, l_q = 0}^{\infty} P_{0,1;l_g,l_q}(Y) P_{l_g,l_q;n_g,n_q}(Y') \tag{2.13}$$

Using these additional equations (10), (12) and (13), we can show that

$$G\left(u_g,u_q;Y+Y'\right)=\sum_{l_g,l_q=0}^{\infty}P_{1,0;l_g,l_q}(Y+Y')u_g^{l_g}u_q^{l_q}$$
$$=\sum_{l_g,l_q=0}^{\infty}\left\{\sum_{n_g,n_q=0}^{\infty}P_{1,0;n_g,n_q}(Y)P_{n_g,n_q;l_g,l_q}(Y')\right\}u_g^{l_g}u_q^{l_q}$$
$$=\sum_{n_g,n_q=0}^{\infty}P_{1,0;n_g,n_q}(Y)\left\{\sum_{l_g,l_q=0}^{\infty}P_{n_g,n_q;l_g,l_q}(Y')u_g^{l_g}u_q^{l_q}\right\}$$
$$=\sum_{n_g,n_q=0}^{\infty}P_{1,0;n_g,n_q}(Y)\left[G\left(u_g,u_q;Y'\right)\right]^{n_g}\left[Q\left(u_g,u_q;Y'\right)\right]^{n_q}$$

$$G\left(u_g,u_q;Y+Y'\right)=G\left[G\left(u_g,u_q;Y'\right),Q\left(u_g,u_q;Y'\right);Y\right] \qquad (2.14)$$

Analogically, probability generating function for quarks is

$$Q\left(u_g,u_q;Y+Y'\right)=Q\left[G\left(u_g,u_q;Y'\right),Q\left(u_g,u_q;Y'\right);Y\right] \qquad (2.15)$$

Recalling (6)-(9) and assuming $P_{1,0;1,0}(\Delta Y)=1$, $P_{0,1;0,1}(\Delta Y)=1$,

$$G\left(u_g,u_q;\Delta Y\right)=u_g+w^{(g)}\left(u_g,u_q\right)\Delta Y+o(\Delta Y) \qquad (2.16)$$

$$Q\left(u_g,u_q;\Delta Y\right)=u_q+w^{(q)}\left(u_g,u_q\right)\Delta Y+o(\Delta Y) \qquad (2.17)$$

Inserting (16), (17) into (14), (15) and replacing Y' with ΔY, we get

$$G\left(u_g,u_q;Y+\Delta Y\right)=$$
$$=G\left[u_g+w^{(g)}\left(u_g,u_q\right)\Delta Y+o(\Delta Y),u_q+w^{(q)}\left(u_g,u_q\right)\Delta Y+o(\Delta Y);Y\right] \qquad (2.18)$$

$$Q\left(u_g,u_q;Y+\Delta Y\right)=$$
$$=Q\left[u_g+w^{(g)}\left(u_g,u_q\right)\Delta Y+o(\Delta Y),u_q+w^{(q)}\left(u_g,u_q\right)\Delta Y+o(\Delta Y);Y\right] \qquad (2.19)$$

Equations (18) and (19) can be expanded into a Taylor series. Assuming $w^{(g)}\left(u_g,u_q\right)\Delta Y+o(\Delta Y)$ as Δu_g and $w^{(q)}\left(u_g,u_q\right)\Delta Y+o(\Delta Y)$ as Δu_q, we obtain

$$G(u_g+\Delta u_g,u_q+\Delta u_q;Y)=G(u_g,u_q;Y)+\frac{\partial G}{\partial u_g}\Delta u_g+\frac{\partial G}{\partial u_q}\Delta u_q+o(\dots)$$

$$\frac{G\left(u_g+\Delta u_g,u_q+\Delta u_q;Y\right)-G(u_g,u_q;Y)}{\Delta Y}=\frac{\partial G}{\partial u_g}\frac{\Delta u_g}{\Delta Y}+\frac{\partial G}{\partial u_q}\frac{\Delta u_q}{\Delta Y}$$

After dividing both sides and letting $\Delta Y\to 0$ the equation will be

$$\frac{\partial G(u_g, u_q; Y)}{\partial Y} = \frac{\partial G}{\partial u_g} w^{(g)}(u_g, u_q) + \frac{\partial G}{\partial u_q} w^{(q)}(u_g, u_q) \qquad (2.20)$$

For quarks analogically

$$\frac{\partial Q(u_g, u_q; Y)}{\partial Y} = \frac{\partial Q}{\partial u_g} w^{(g)}(u_g, u_q) + \frac{\partial Q}{\partial u_q} w^{(q)}(u_g, u_q) \qquad (2.21)$$

(20) and (21) are the forward Kolmogorov equations for the generating functions of the transition probabilities $P_{m_g,m_q;n_g,n_q}(Y)$. But for solving our problem it's necessary to get corresponding backward Kolmogorov equations, which immediately follows from inserting (16), (17) into (14), (15) and letting $Y \to \Delta Y$.

$$G(u_g, u_q; \Delta Y + Y') = G[G(u_g, u_q; Y'), Q(u_g, u_q; Y'); \Delta Y] =$$

$$= G(u_g, u_q; Y') + w^{(g)}[G(u_g, u_q; Y'), Q(u_g, u_q; Y')]\Delta Y + o(\Delta Y)$$

After dividing both sides by $\Delta Y \to 0$ we get

$$\frac{\partial G}{\partial Y} = w^{(g)}[G(u_g, u_q; Y), Q(u_g, u_q; Y)] \qquad (2.22)$$

$$\frac{\partial Q}{\partial Y} = w^{(q)}[G(u_g, u_q; Y), Q(u_g, u_q; Y)] \qquad (2.23)$$

For both cases of Kolmogorov equations there are initial conditions that are given by

$$G|_{Y=0} = u_g, Q|_{Y=0} = u_q \qquad (2.24), (2.25)$$

Using (6), (7) our equations become

$$\frac{\partial G}{\partial Y} = -AG - BG + AG^2 + BQ^2 \qquad (2.26)$$

$$\frac{\partial Q}{\partial Y} = -\tilde{A}Q + \tilde{A}QG \qquad (2.27)$$

In an intuitive approach our problem can be reformulated directly in terms of the transition probability functions (normalized exclusive cross sections of producing n_g gluons and n_q quarks) without going through the generating function.

$$P_{1,0;n_g,n_q}(Y) \equiv \frac{\sigma(g \to n_g + n_q)}{\sigma_{total}}$$

$$P_{0,1;n_g,n_q}(Y) \equiv \frac{\sigma(q \to n_g + n_q)}{\sigma_{total}}$$

We can give certain initial conditions, such as presence of 1 gluon and no quark or of 1 quark and no gluon at $Y = 0$, to compose the probability for a gluon and quark to produce n_g gluons and n_q quarks in the interval $(Y, Y + \Delta Y)$. Taking into account that only 3 processes are allowed, and at the same time nothing can happen, it follows for a gluon jet

$$P_{1,0;n_g,n_q}(Y + \Delta Y) = \tilde{A}\Delta Y P_{1,0;n_g-1,n_q}(Y)\cdot n_q + A\Delta Y P_{1,0;n_g-1,n_q}(Y)\cdot\left(n_g - 1\right) + \\ + B\Delta Y P_{1,0;n_g+1,n_q-2}(Y)\cdot\left(n_g + 1\right) + \\ + \left(1 - \tilde{A}\Delta Y n_q - A\Delta Y n_g - B\Delta Y n_g\right)P_{1,0;n_g,n_q}(Y) + o(Y) \quad (2.28)$$

and for a quark jet

$$P_{0,1;n_g,n_q}(Y + \Delta Y) = \tilde{A}\Delta Y P_{0,1;n_g-1,n_q}(Y)\cdot n_q + A\Delta Y P_{0,1;n_g-1,n_q}(Y)\cdot\left(n_g - 1\right) + \\ + B\Delta Y P_{0,1;n_g+1,n_q-2}(Y)\cdot\left(n_g + 1\right) + \\ + \left(1 - \tilde{A}\Delta Y n_q - A\Delta Y n_g - B\Delta Y n_g\right)P_{0,1;n_g,n_q}(Y) + o(Y) \quad (2.29)$$

Dividing by ΔY and letting $\Delta Y \to 0$ we obtain the system of differential equations.

$$\frac{dP_{1,0;n_g,n_q}(Y)}{dY} = \left(-\tilde{A}n_q - An_g - Bn_g\right)P_{0,1;n_g,n_q}(Y) + \\ + \tilde{A}P_{0,1;n_g-1,n_q}(Y)\cdot n_q + AP_{0,1;n_g-1,n_q}(Y)\cdot\left(n_g - 1\right) + \\ + BP_{0,1;n_g+1,n_q-2}(Y)\cdot\left(n_g + 1\right) \quad (2.30)$$

If we are only interested in the gluon exclusive cross section both in a gluon- or a quark-jet, we obtain

$$\frac{dP_{1,0;n_g,0}(Y)}{dY} = -(B + A)n_g P_{1,0;n_g,0}(Y) + A\left(n_g - 1\right)P_{1,0;n_g-1,0}(Y) \quad (2.31)$$

$$\frac{dP_{0,1;n_g,1}(Y)}{dy} = -\tilde{A}P_{0,1;n_g,1}(Y) - (B + A)n_g P_{0,1;n_g,1}(Y) + \\ + \tilde{A}P_{0,1;n_g-1,1}(Y) + A\left(n_g - 1\right)P_{0,1;n_g-1,1}(Y) \quad (2.32)$$

3. Explicit solutions in particular cases

Finding the explicit solution of the cross section is not easy. However, approximate solutions can be obtained for particular cases, which can help us to understand the general problem.

We consider that $B = 0, A \neq \tilde{A} \neq 0$. It means we forbid gluons to split into quark-antiquark pair (or absence of flavours in theory). Equations for the cross sections for n_g gluons in the gluon- and quark- jet will be

$$\frac{dP_{1,0;n_g,0}(Y)}{dY} = -An_g P_{1,0;n_g,0}(Y) + A(n_g - 1)P_{1,0;n_g-1,0}(Y) \tag{3.1}$$

$$\frac{dP_{0,1;n_g,1}(Y)}{dy} = -\tilde{A}P_{0,1;n_g,1}(Y) - An_g P_{0,1;n_g,1}(Y) +$$

$$+\tilde{A}P_{0,1;n_g-1,1}(Y) + A(n_g - 1)P_{0,1;n_g-1,1}(Y) \tag{3.2}$$

with initial conditions

$$P_{1,0;n_g,0}(0) = \delta_{1n_g} \tag{3.3}$$

$$P_{0,1;n_g,1}(0) = \delta_{0n_g} \tag{3.4}$$

From (3.1) and (3.3) we get for gluon jet

$$P_{1,0;1,0}(Y) = e^{-AY} \tag{3.5}$$

$$P_{1,0;n_g,0}(Y) = e^{-AY}(1 - e^{-AY})^{n_g-1} \tag{3.6}$$

The corresponding generating function is

$$G = \sum_{n_g=1}^{\infty} u_g^{n_g} P_{1,0;n_g,0}(Y) = \frac{u_g e^{-AY}}{1 - u_g(1 - e^{-AY})} \tag{3.7}$$

Moreover,

$$\langle n_g \rangle = \frac{\partial G}{\partial u_g} \Big|_{u_g=1} = e^{AY} \tag{3.8}$$

Therefore, normalized cross section is

$$\frac{\sigma_{n_g,0}^{(g,0)}}{\sigma_{total}} \equiv P_{1,0;n_g,0}(Y) = \frac{1}{\langle n_g \rangle}\left(1 - \frac{1}{\langle n_g \rangle}\right)^{n_g-1} \tag{3.9}$$

And for quark jet, where $\mu = \frac{\tilde{A}}{A}$

$$P_{0,1;0,1}(Y) = e^{-\tilde{A}Y} \tag{3.10}$$

$$P_{0,1;n_g,1}(Y) = \frac{\mu(\mu+1)\ldots(\mu+n_g-1)}{n_g!} e^{-\tilde{A}Y}(1-e^{-AY})^{n_g} \tag{3.11}$$

The corresponding generating function will be given by

$$Q = \sum_{n_g=0}^{\infty} u_g^{n_g} u_q P_{0,1;n_g,1}(Y) = u_q \left(\frac{e^{-AY}}{1-u_g(1-e^{-AY})} \right)^{\mu} \tag{3.12}$$

and average gluon multiplicity (with $u_q = 1$)

$$\langle n_g \rangle = \frac{\partial Q}{\partial u_g} \Big|_{u_g=1} = \mu(e^{AY}-1) \tag{3.13}$$

The normalized exclusive cross section

$$\frac{\sigma_{n_g,0}^{(0,q)}}{\sigma_{total}} \equiv P_{0,1;n_g,1}(Y) =$$

$$= \frac{\mu(\mu+1)\ldots(\mu+n_g-1)}{n_g!} \left(\frac{\langle n_g \rangle}{\langle n_g \rangle + \mu} \right)^{n_g} \left(\frac{\mu}{\langle n_g \rangle + \mu} \right)^{\mu} \tag{3.14}$$

The second correlation moment is given by

$$f_2 = Q'' \big|_{u_g=1} - \left(Q' \big|_{u_g=1} \right)^2 = \mu(e^{AY}-1)^2 > 0 \tag{3.15}$$

Let’s analyze out results. (3.14) is a Polya-Eggenberger distribution where μ is half integrated in the limit of $N_c \rightarrow \infty$. In Two Stage model it is used for description the cascade stage. (3.9) is a Furry-Yule distribution which corresponds to a (3.14) with $\mu = 1$.

The experimental data of e^+e^- annihilation with energy lower than 9 GeV shows us $f_2 < 0$. To deal with this problem we added binomial distribution for hadronization stage.

4. Sources

[1] QCD JETS AS MARKOV BRANCHING PROCESSES - A. Giovannini

[2] Элементы теории марковских процессов – Баруча-Рид А.Т.

[3] Interest_Kokoulina_file_project_261.pdf (jinr.ru)