Numerical Methods in Theory of Topological Solitons

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Abstract

In this paper we revise the theory of ϕ^4 kinks and find the static analytical solution for a standard potential. Then we consider potentials for which the resulting field equations can't be solved analytically and thus we approach them using finite difference approximations and Newton's method for nonlinear systems of equations.

1 Introduction

Solitons are wave packets that preserve their shape while propagating with constant velocity and arise as non-trivial solutions to nonlinear field equations. It is often infeasible to find analytical solutions, so we have to recur to numerical methods. However, before we delve into how we can find solutions numerically, we will revise the theory of ϕ^4 kinks in 1 + 1 dimensions and consider how we can find the analytical solution for a specific potential. The scalar field $\phi(x,t)$ has the associated Lagrangian density

$$Ł = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi),$$

which gives the corresponding field equation

$$\partial_\mu\partial^\mu\phi + \frac{dV}{d\phi} = 0.$$

If we consider a static solution, we can omit the time derivatives to get

$$\frac{d^2\phi}{dx^2} = \frac{dV}{d\phi}.$$

Now we study solutions for the standard potential

$$V = \frac{1}{2}(1-\phi^2)^2,$$

with 2 vacua $\phi(x) = \pm 1$. The field equation becomes

$$\frac{d^2\phi}{dx^2} = -2(1-\phi^2)\phi,$$

from which, after some derivation using Noether's theorem, we get that

$$\frac{d\phi}{dx} = 1-\phi^2,$$

which gives the kink and the corresponding anti-kink solutions

$$\phi_K = \tanh(x-\alpha), \quad \phi_{\overline{K}} = \tanh(x-\beta).$$

We can obtain the dynamic solution by applying a Lorentz boost to get

$$\phi(x,t) = \tanh(\gamma(x-vt)).$$

2 Numerical methods for static solutions

Unfortunately, it is not always possible to get analytical solutions. For

$$V(\phi) = (1-\epsilon)(1-\cos(\phi)) + \frac{\epsilon\phi^2}{8\pi^2}(\phi-2\pi)^2,$$

the resulting (static) field equation is

$$\frac{d^2\phi}{dx^2} = (1-\epsilon)\sin(\phi) + \frac{\epsilon}{8\pi^2}(4\phi^3 - 12\pi\phi^2 + 8\pi^2\phi),$$

which can only be solved numerically. We can approximate ϕ in $N+1$ points

$$\boldsymbol{\phi} = \begin{pmatrix}\phi_0 & \phi_1 & ... & \phi_N\end{pmatrix}^T, \quad \phi_i \approx \phi(x_i),$$

where

$$x_i = (x_0 + \frac{x_N - x_0}{N} i.)$$

and ϕ_0 and ϕ_N are known since we set them as the initial conditions. Hence, the vector of unknown variables is

$$\boldsymbol{\phi'} = \begin{pmatrix}\phi_1 & \phi_2 ... & \phi_{N-1}\end{pmatrix}^T.$$

Using finite differences, we can approximate the field equation with

$$\frac{\phi_{n-1} - 2\phi_n + \phi_{n+1}}{h^2} = (1-\epsilon)\sin(\phi_n) + \frac{\epsilon}{8\pi^2}(4\phi_n^3 - 12\pi\phi_n^2 + 8\pi^2\phi_n),$$

for $n = 1$ to $n = N-1$. We have a nonlinear system of $N-1$ equations for $N-1$ unknown variables, which can be solved using Newton's method. Setting

$$f_n = \phi_{n-1} - 2\phi_n + \phi_{n+1} - h^2((1-\epsilon)\sin(\phi_n) + \frac{\epsilon}{8\pi^2}(4\phi_n^3 - 12\pi\phi_n^2 + 8\pi^2\phi_n))$$

and

$$\boldsymbol{f}(\boldsymbol{\phi'}) = \begin{pmatrix} f_1(\boldsymbol{\phi'}) & f_2(\boldsymbol{\phi'}) & \dots & f_{N-1}(\boldsymbol{\phi'}) \end{pmatrix}^T$$

we get the Jacobian

$$\boldsymbol{J} = \begin{bmatrix} \alpha_1 & 1 & 0 & 0 & \dots & 0 \\ 1 & \alpha_2 & 1 & 0 & \dots & 0 \\ 0 & 1 & \alpha_3 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \alpha_{N-2} & 1 \\ 0 & 0 & \dots & 0 & 1 & \alpha_{N-1} \end{bmatrix},$$

where

$$\alpha_n = -2 - h^2((1-\epsilon)\cos(\phi_n) + \frac{\epsilon}{8\pi^2}(12\phi_n^2 - 24\pi\phi_n + 8\pi^2)).$$

Starting from an initial guess $\boldsymbol{\phi'}_0 = \mathbf{0}$, we can compute better guesses iteratively with

$$\boldsymbol{\phi'}^{(i+1)} = \boldsymbol{\phi'}^{(i)} - [\boldsymbol{J}^{(i)}]^{-1}\boldsymbol{f}(\boldsymbol{\phi'}^{(i)}).$$

Setting

$$\phi(-\infty) = 0, \quad \phi(\infty) = 2\pi$$

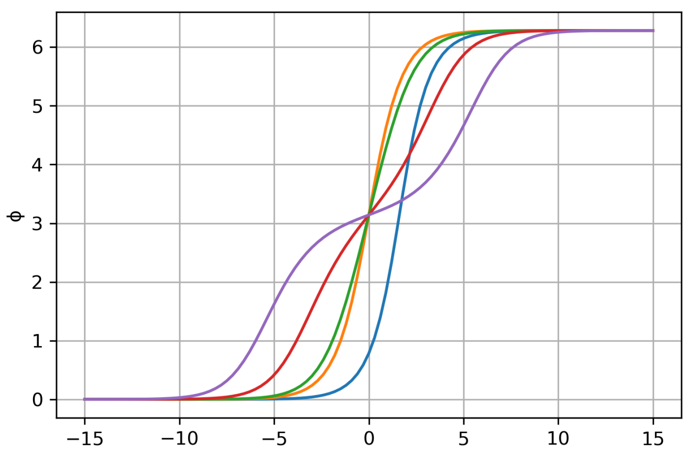
and

$$x_0 = -15, \quad x_N = 15,$$

we can approximate the initial conditions as

$$\phi(-15) = 0, \quad \phi(15) = 2\pi.$$

The resulting approximation for $N = 100$, 400 iterations and $\epsilon = 0, 0.5, 1.5, 2.5, 2.6$ is shown below:



Now we consider the solution for the potential

$$V(\phi) = \frac{1}{2}(\phi^2 - a^2)^2(\phi^2 - b^2)^2.$$

The associated field equation is

$$\frac{d^2\phi}{dx^2} = 2\phi(\phi^2 - a^2)(\phi^2 - b^2)(-a^2 - b^2 + 2\phi^2).$$

Similarly as with the previous potential, we use a finite difference approximation of the field equation on $N+1$ points to get a system of N-1 nonlinear equations

$$f_n = \phi_{n-1} - 2\phi_n + \phi_{n+1} - 2h^2\phi_n(\phi_n^2 - a^2)(\phi_n^2 - b^2)(-a^2 - b^2 + 2\phi_n^2),$$

for $n = 1$ to $n = N - 1$, which we will solve using Newton's method. The resulting Jacobian matrix is

$$\boldsymbol{J} = \begin{bmatrix} \beta_1 & 1 & 0 & \dots \\ 1 & \beta_2 & 1 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & \beta_{N-1} \end{bmatrix},$$

where

$$\beta_n = -2 + 2h^2(a^4(b^2 - 3\phi_n^2) + a^2(b^4 - 12b^2\phi_n^2 + 15\phi_n^4) - 3b^4\phi_n^2 + 15b^2\phi_n^4 - 14\phi_n^6).$$

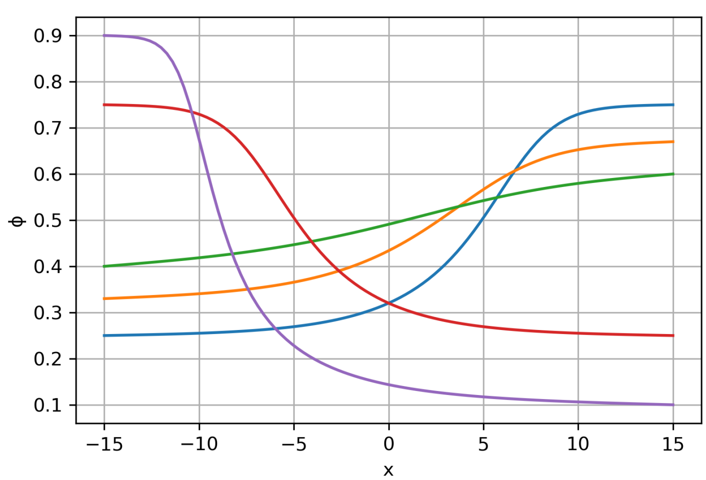
We set $b = 1 - a$ and the initial conditions as

$$\phi(-\infty) = a, \quad \phi(\infty) = b$$

and $x_0 = -15$, $x_N = 15$. We can then approximate the initial conditions as

$$\phi(-15) = a, \quad \phi(15) = b.$$

The approximation of $\phi(x)$ for N=100, 300 iterations and $a = 0.25, 0.33, 0.4, 0.75, 0.9$ is below:



References

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