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BOGOLUBOV LABORATORY OF THEORETICAL PHYSICS

Final report on the INTEREST programme

Numerical methods in theory of topological solitons

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Abstract

In this paper we consider static topological solitonic solutions for $O(3)$ sigma model. In the beginning some theoretical references are noted, such as topological nature of solitons and isomorphism between domain and target spaces. The feature of equivalence of north and south pole stereographic projections was also discussed. Then several cases of solutions were sufficiently delailed and analysed using numerical methods in C++ and visualised with graphing software.

1 Introduction

The possibility of existance of spacially localized solitonic solution in scalar field theory with classical Lagrangian density, e.g.

$$L = \frac{1}{4}\partial_\mu \varphi^a \partial^\mu \varphi^a - U(|\varphi|) \tag{1}$$

is defined by the number of spacial dimension d of that space. Derrick's theorem states[3] that if we deal with model in d=2 spatial dimensions, which may include a set of N scalar fields, it will be impossible to construct scale invariant (i.e. stable with regard to changing its characteristic size) solitonic solution. However, it does not mean the solitons are excluded; the subtlety related to the possible choice of the vacuum boundary conditions allows us to construct topologically nontrivial planar solitons.

One of such cases is sigma model – a nonlinear scalar field theory, where the field takes values in a target space which is a curved Riemannian manifold, usually with a large symmetry. The simplest example is the $O(3)$ sigma model, in which the target space is the unit 2-sphere, S^2. $O(3)$ in the model name stands for group of rotational symmetry of a sphere, the "*sigma*" refers to the fact that the model is sometimes formulated in terms of fields φ_1, φ_2, σ and σ, obviously, can be derived from two other components due to their location on unit sphere of constant radius.

This field model was originally designed as a simplified model of strong interactions between nucleons and π-mesons[4]. It is also known in theory of condensed matter as continuum approximation of the 2-dimensional isotropic Heisenberg ferromagnet[5]. The topological solutions of this model were discovered by Belavin and Polyakov in 1974 during their work on Metastable states of two-dimensional isotropic ferromagnets[6].

2 General theory

2.1 Isomorphism between spaces $\mathbb{C}P^1$ and S^2

Let us now consider the triplet of real scalar fields $\varphi^a = (\varphi_1, \varphi_2, \varphi_3)$ restricted to the unit sphere S^2 via the constraint

$$\varphi^a \cdot \varphi^a = 1 \tag{2}$$

and choose potential energy in Lagrangian (1) in following way:

$$L = \frac{1}{4}\partial_\mu \varphi^a \partial^\mu \varphi^a + \lambda(1 - \varphi^a \cdot \varphi^a) \tag{3}$$

where λ is the Lagrange multiplier. An exceptional property of the sphere S^2 is that it admits a complex structure, i.e., $S^2 = \mathbb{C}P^1$. Indeed, the usual stereographic projection from the sphere

S^2 to the complex plane allows us to reformulate the model in terms of the complex variable (see Figure 1): Stereographic projection from the north pole **N** can be obtained by using simple geometrical sphere equation in field components, already mentioned as (1) and straight line equation in $\mathbb{R}^3(x, y, z)$ between two points N(0,0,1) and W(Re(W),Im(W),0):

$$\frac{1-z}{1} = \frac{x}{Re(W)} = \frac{y}{Im(W)} \tag{4}$$

Now combining (2) and (4) we get fields $W, \overline{W}$, which are inhomogeneous coordinates on the one-dimensional projective space $\mathbb{C}P^1$:

$$W = \frac{\varphi_1 + i\varphi_2}{1-\varphi_3} \tag{5}$$

Then the inverse transformations are:

$$(\varphi_1, \varphi_2, \varphi_3) = \left(\frac{W+\overline{W}}{1+W\overline{W}}, i\frac{\overline{W}-W}{1+W\overline{W}}, \frac{1-W\overline{W}}{1+W\overline{W}}\right) \tag{6}$$

And, obviously, Lagrangian (3) can be rewritten in following way:

$$L = \frac{\partial_\mu W \partial^\mu \overline{W}}{\left(1+W\overline{W}\right)^2} \tag{7}$$

In terms of holomorphic derivatives energy of static configuration takes form

$$E = \frac{|W_z|^2 + |W_{\overline{z}}|^2}{\left(1+|W|^2\right)^2} \tag{8}$$

The last identity will be used further.

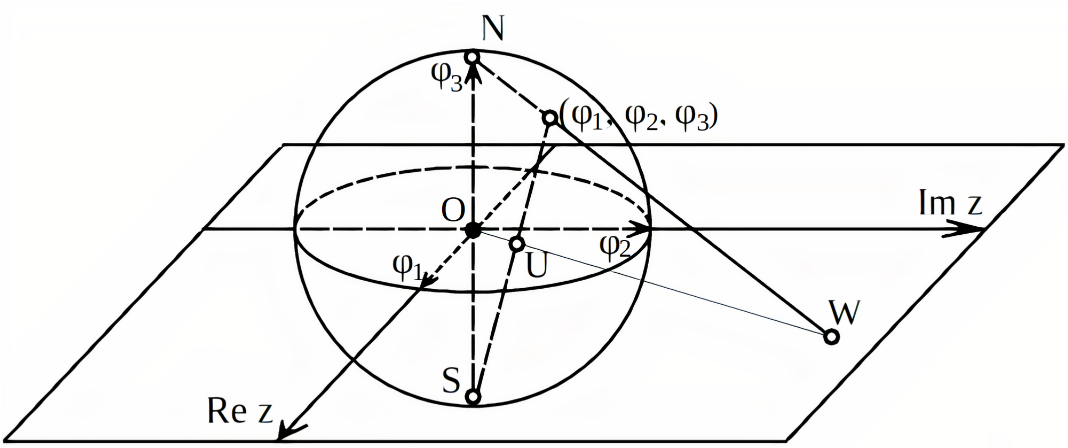

Figure 1: Stereographic projection $S^2 \mapsto \mathbb{R}^2 = \mathbb{C}$

2.2 Equivalence between north and south pole stereographic projections

It deserves to be acknowledged that this model also, clearly, maintains stereographic projection from the south pole **S** and, moreover, it does not change the form of Lagrangian density (7), (8).

Let us again consider Figure 1

New variable U now stands for south pole stereographic projection and its value can be easily found with the help of simple geometrical relations: cross section of unit sphere by plane (**NWO**) contains unit circle with centre in point **O** and two triangles: $\triangle$ **NOW** and $\triangle$ **N**Φ_i**S**, where Φ_i is considered point of triplet of fields on the unit sphere. Apparent ratio between absolute values of U and W is $|U| = |W|^{-1}$. It is clear, that they have the same argument θ, so we can easily construct them:

$$U = \frac{1}{|W|} e^{i\theta},\ W = |W|\, e^{i\theta} \Rightarrow U = \overline{W}^{-1} \tag{9}$$

After some calculations it can be figured out that Lagrangians of north and south stereographic projections have equivalent form and, ergo, energies of static configurations are equal to each other, i.e.:

$$E = \frac{|W_z|^2 + |W_{\overline{z}}|^2}{\left(1 + |W|^2\right)^2} = \frac{|U_z|^2 + |U_{\overline{z}}|^2}{\left(1 + |U|^2\right)^2} \tag{10}$$

In the next chapter of the report it will be shown that changing the point of consideration of solitonic configuration from north to south and vice versa will not effect any of its parameters on complex plane.

2.3 Topological nature of solitonic solutions

Let us now look at the integral energy of static configuration

$$E = \frac{1}{2} \int \partial_k \varphi^a \partial^k \varphi^a d^2x \tag{11}$$

and also at its alternative forms (8), (10). It is clear, that vacuum state of energy $\varphi_{vac} = (0, 0, 1)$ is approached on the domain space $\mathbb{R}^2$ (or $\mathbb{C}$) at points that lie in infinity. That is why we can identify them and compactify the domain space from $\mathbb{R}^2$ to S^2. Then the field of the $O(3)$ model becomes a map $\varphi : S^2 \mapsto S^2$ from physical space to target space, which is classified by the homotopy group $\Pi_2\left(S^2\right) = \mathbb{Z}$. This property implies that each field configuration is characterized by an integer topological charge Q

$$Q = \frac{1}{4\pi} \int \vec{\varphi} \cdot (\partial_x \vec{\varphi} \times \partial_y \vec{\varphi}) d^2x \tag{12}$$

The charge is equal to the number of solitons on the $\mathbb{C}$ plane. After that, considering obvious integral inequality

$$\int \left(\partial_i \vec{\varphi} \pm \varepsilon_{ij} \vec{\varphi} \times \partial_j \vec{\varphi}\right) \cdot \left(\partial_i \vec{\varphi} \pm \varepsilon_{ik} \vec{\varphi} \times \partial_k \vec{\varphi}\right) d^2x \geq 0 \tag{13}$$

and using formulae (11), (12) we get the Bogomolny bound $E \geq 2\pi\,|Q|$, the lower bound on the energy in terms of the number of solitons. It becames equality, when the factors in expression (13) are equal to zero. If we reformulate them in terms of change of variable (5), (6), they turn into the equivalent of Cauchy-Reimann equations:

$$W_{\overline{z}} = 0 \quad and \quad W_z = 0 \tag{14}$$

for "+" and "-" signs in (13) respectively. Then the general expression for n-solitonic solution is a holomorphic map

$$W(z) = \frac{P_k(z)}{Q_m(z)} \tag{15}$$

where P_k, Q_m are polynomials and n = max(k,m).

3 Solution

The problem of constructing plots of field components and energy density did not require higly sophisticated numerical methods. All calculations were done in C++. To facilitate process of working with complex numbers library `<complex>` was used. The only nontrivial issue was calculating complex variables in (8) and (10). In this work Cauchy's differentiation formula was used to meet the challenge:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)dz}{(z - z_0)^{n+1}} \tag{16}$$

It can be turned into code and calculate derivative `derres` with high accuracy:

```
double n = pow(N,-1);
     double phi = (2 * M_PI*n);
       complex<double> z1;
       double R = 0.01;
       complex<double> z0(real(z) + R, imag(z));

 for (int j = 0; j <=(N + 1); j++)
{
 complex<double> z0(real(z) + R * cos(phi*j), imag(z) + R * sin(phi*j));
 complex<double> z1(real(z) + R * cos(phi*(j+1)), imag(z) + R * sin(phi*(j+1)));
 delta_z = z1 - z0;
 derres = derres + U(z0)*pow(z0 - z, -2)*delta_z;
}
       derres = (-I / (2 * M_PI))*derres;
```

A circle of R radius with the centre in z_0 point is chosen as a closed circuit. Other aspects of full code are too simple and not worth discussing.

4 Results

4.1 Equivalence between N and S stereographic projections

As it was said in part 2.2 of the report, changing point of viev from one pole to the opposite does not change the solution. Let us check it on some soliton configuration now. Consider holomorphic map of degree 2:

$$W(z) = \frac{\lambda_1 e^{i\theta_1}}{z - a_1} + \frac{\lambda_2 e^{i\theta_2}}{z - a_2} \tag{17}$$

Where $\lambda_1 = 1.3$, $\theta_1 = 1$, $a_1 = i$ and $\lambda_2 = 1.0$, $\theta_2 = 2$, $a_2 = 1$. Now let us use identity (9):

$$U(z) = \frac{1}{\frac{\lambda_1 e^{-i\theta_1}}{\bar{z} - \bar{a}_1} + \frac{\lambda_2 e^{-i\theta_2}}{\bar{z} - \bar{a}_2}} \tag{18}$$

And compare graphs of energy density:

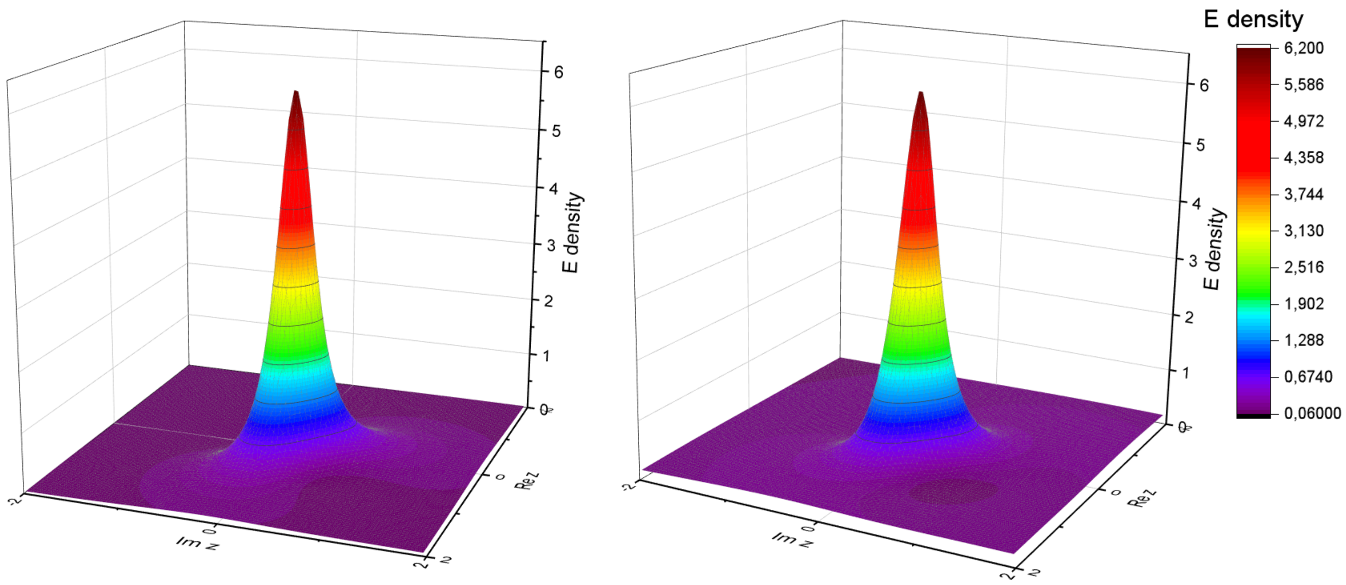

Figure 4.1: Application of formula (10) to configurations (17) and (18)

As it can be seen from Figure 4.1, the graphs are absolutely identical. Solitons are also kind of merged together due to the properties of multisolitonic solutions.

4.2 Building field components of configuration

Let us consider a group of Q=8 solitons on a complex plane. The corresponding holomorphic map is:

$$W(z) = \frac{1}{\frac{1}{z} + \frac{1}{z+\frac{1}{2}-i} + \frac{1}{z-\frac{1}{2}-i} + \frac{1}{z-1} + \frac{1}{z+1} + \frac{1}{z+\frac{3}{2}+i} + \frac{1}{z+\frac{3}{2}-i} + \frac{1}{z+2i}} \tag{19}$$

Using transformations (6) we can build graphs of field components. See the parameters on figure 4.2.

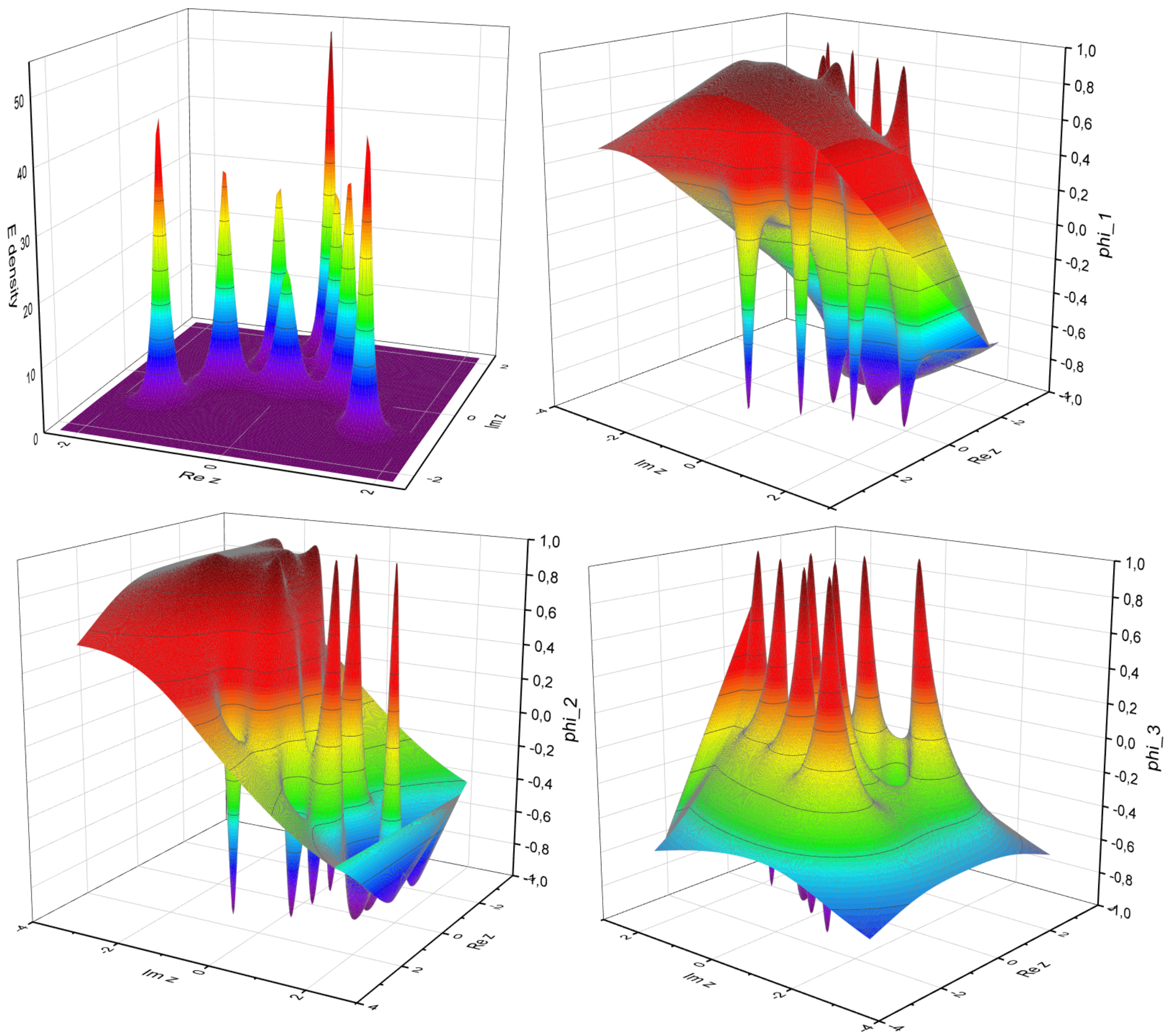

Figure 4.2: Parameters of config. (19).
Upper left plot – Energy density, upper right – φ_1 component, bottom left – φ_2 component, bottom right – φ_3 component.

4.3 Configuration of two solitons with different parameters

A general formula of Q=2 configuration is

$$W(z) = \frac{(z-a)(z-b)}{(z-c)(z-d)} \tag{20}$$

Where $a, b, c, d \in \mathbb{C}$.

Here are three cases of different combinations:

- The first case: a = (1,1); b = (1,-1); c = (-1,1); d = (-1,-1);
- The second case: a = (0.7,0.7); b = (-0.7,0.7); c = (-2,-2) ; d = (0.7,-0.7) ;
- The third case: a = (-2,0); b =(0,2); c = (1,0.1); d = (1,-0.1).

See the results of graphing energy density on figure 4.3.

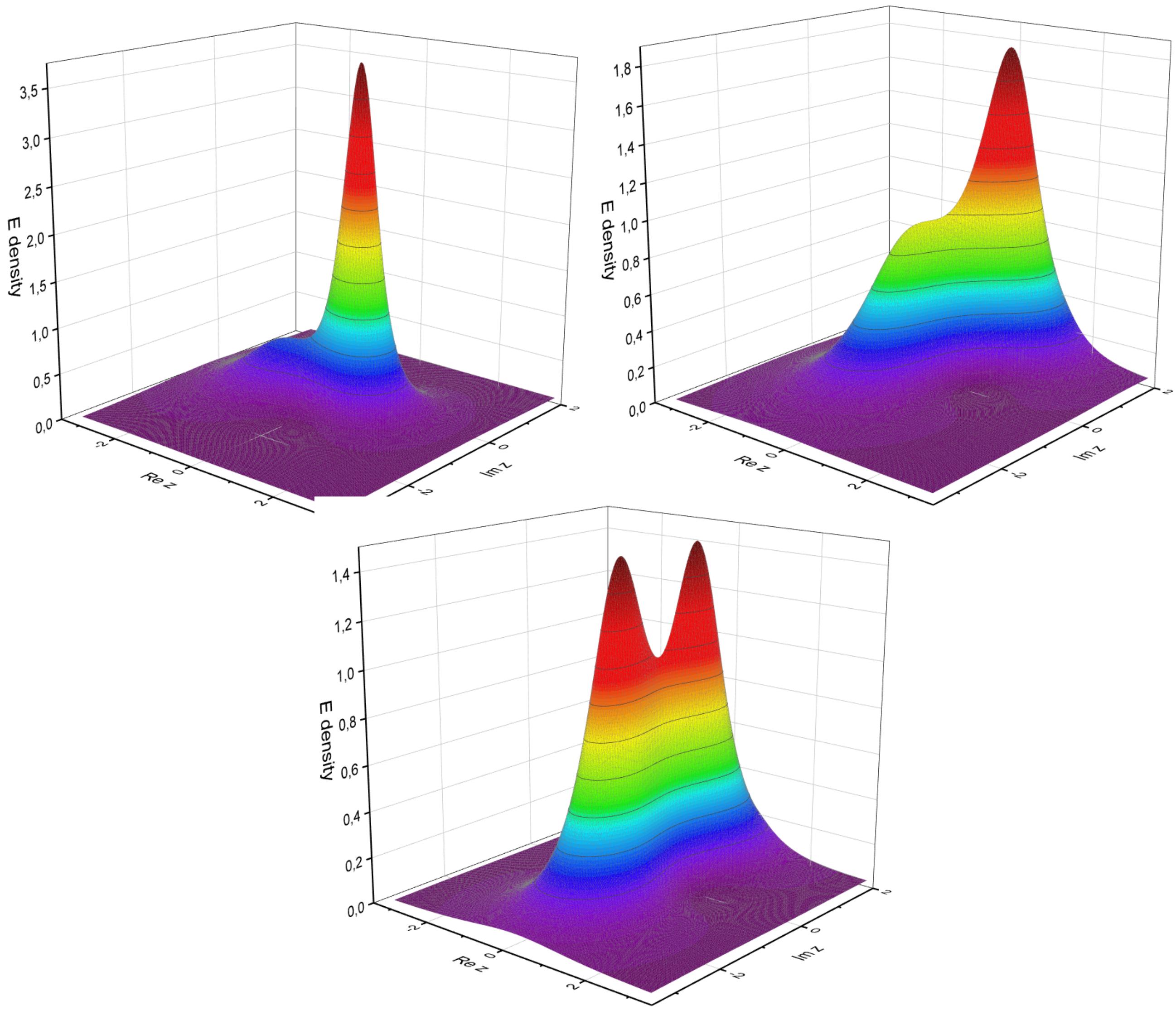

Figure 4.3: Energy density of config. (20).
Upper left plot – the third case, upper right – the second case, bottom plot – the first case

4.4 Changing position of solitons

Let once again look at formula (19). It can be easily seen both from theory and graph 4.2, that numbers near z in each of the fractions are the positions of corresponding lumps of energy. Changing them in following way:

$$W(z) = \frac{1}{\frac{1}{z+4} + \frac{1}{z+3} + \frac{1}{z+2} + \frac{1}{z+1} + \frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-2} + \frac{1}{z-3}} \tag{21}$$

Gives us configuration of 8 equidistant solitons, aligned along X axis. See figure 4.4:

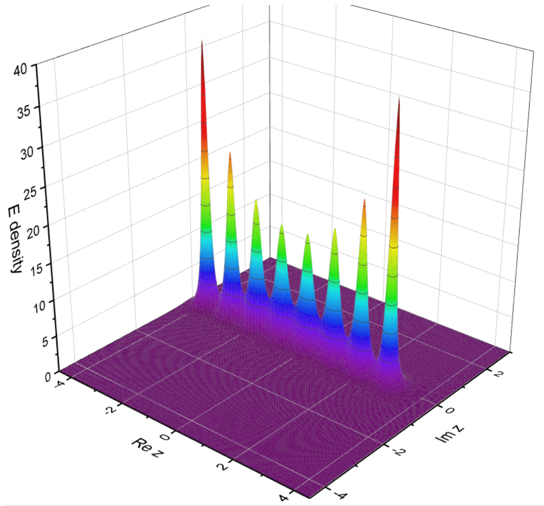

Figure 4.4: Energy density of aligned configuration

As it was mentioned in part 4.1 of the report, their shape is not equal due to features of multisolitonic solutions. This issue can be fixed by relocating them to infinite distance from each other. In such a case they all will be the same form and shape as a lone soliton configuration Q=1 with set parameters.

5 Conclusion

To conclude, $O(3)$ sigma model – a nonlinear scalar field theory was outlined and its key points were analysed. Such of its features as constructing soliton configurations from the point of viev of topology and isomorphism between spaces $\mathbb{C}P^1$ and S^2 were discussed. Also the property of equivalence of north and south pole stereographic projections was geometrically derived, proven and later used. Several graphs of different solitonic configurations were obtained via numerical methods (C++) and data visualising software.

References

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