JOINT INSTITUTE FOR NUCLEAR RESEARCH
Bogoliubov Laboratory of Theoretical Physics

FINAL REPORT ON INTEREST PROGRAMME

Numerical methods in theory of topological solitons

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1 Abstract

In this paper, we consider four different field theory models. We investigate various potential models and study their properties. The main aim is to obtain and analyze elementary solutions of the equations of motion in these models - kinks. We obtain these solutions using numerical methods in C ++. We consider stationary solutions to our equations.

2 Introduction

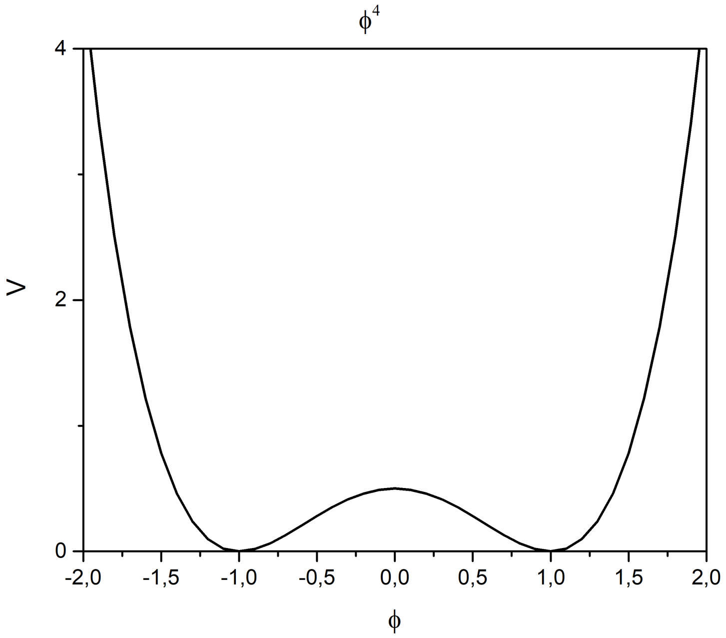
A kink is a solution to field equations in some field theories in $1+1$ dimensions, interpolating between two vacuums as the spatial coordinate changes from $-\infty$ to $+\infty$. The kink is the simplest topological soliton.

In our work, we consider four models of field theory. To find a solution in the form of a kink, it is necessary to determine the vacuum states of our models, which are found as minima of the interaction potential functions. We choose two neighboring vacuum states, with the help of which we set the boundary conditions for elementary solutions of the equations of motion of our models. Now we have a well-posed boundary value problem, which we write using the difference scheme for the grid. Further, using C ++, we have the opportunity to obtain the desired solution with good accuracy and represent it using graphs.

3 ϕ^4 model

We begin our consideration with the ϕ^4 model, in which the potential can be written in the form

$$V(\phi) = \frac{1}{2}\left(1 - \phi^2\right)^2.$$



Then, using the general formula for writing the Lagrangian

$$L = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - V(\phi),$$

we obtain the Lagrangian for this theory and represent it using the expression

$$L = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}\left(1 - \phi^2\right)^2.$$

In order to obtain the equation of motion, we write down the Lagrange equation

$$\frac{\partial}{\partial x_\mu}\left(\frac{\delta L}{\delta\left(\partial_\mu \phi\right)}\right)=\frac{\delta L}{\delta \phi}.$$

After substituting the Lagrangian of our theory, we obtain the desired equation of motion

$$\partial_\mu^2 \phi = 2\phi(1-\phi^2).$$

Since our task conditionally has an infinite number of solutions, we need to select one. This can be done by writing the boundary conditions that arise from the constraints and properties of our model. Then, in the final form, we can write our problem in the form

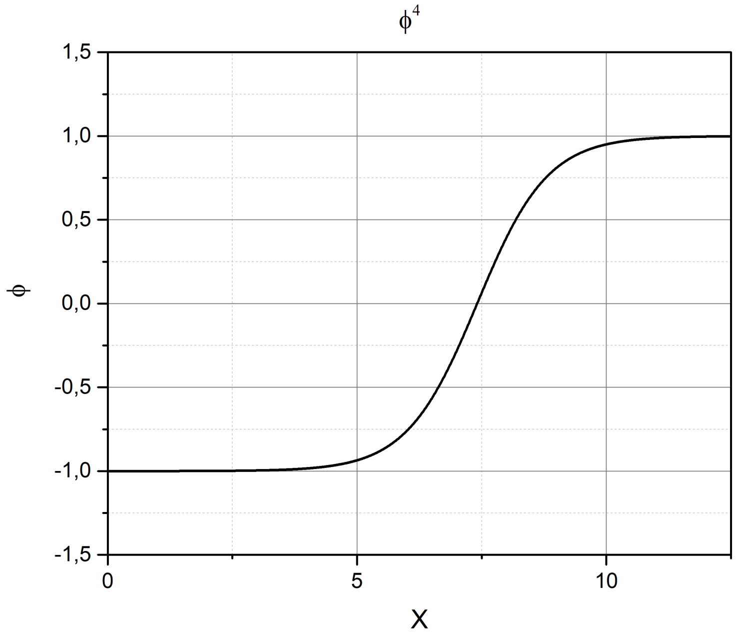
$$\begin{cases} \partial_\mu^2 \phi = 2\phi(1-\phi^2), \\ \phi(-\infty) = -1, \\ \phi(\infty) = 1. \end{cases}$$

In order to obtain a numerical solution in the form of a kink, we need to draw up a difference scheme for our problem

$$\begin{cases} \frac{\phi_{n-1}-2\phi_n+\phi_{n+1}}{h^2} = 2\phi_n(1-\phi_n^2), \\ \phi_0 = -1, \\ \phi_N = 1, \end{cases}$$

where $n \in (0, N)$ and h is the grid step

Solving this boundary value problem numerically, we obtain a solution in the form of a kink, which can be represented using the graph



4 ϕ^6 model

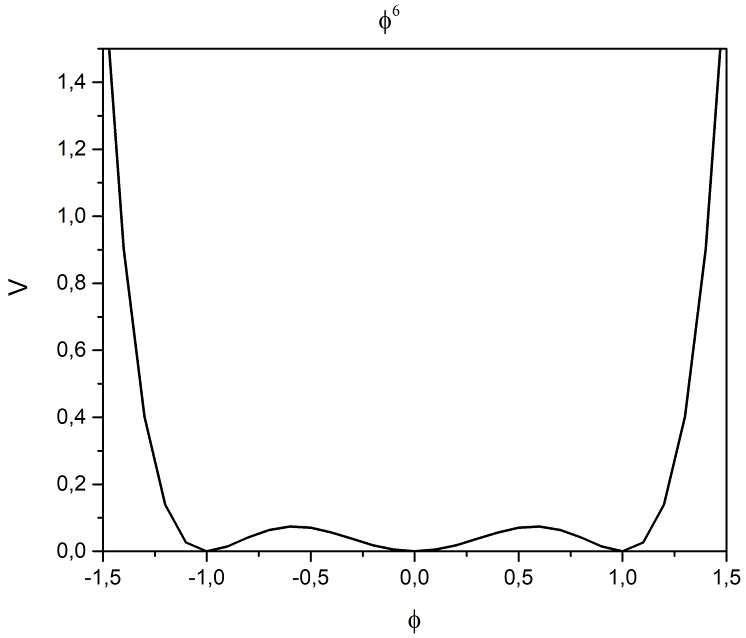
Now let us consider another model, in which the field is already included in the sixth degree. We write the potential of this model in the form

$$V(\phi) = \frac{1}{2}\phi^2\left(1-\phi^2\right)^{2.}$$

Using the already known formula, we represent the Lagrangian of our model in the form

$$L = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}\phi^2\left(1-\phi^2\right)^2.$$

Substituting our Lagrangian into the Lagrangian equation, and transforming the corresponding expression, we obtain the equation of motion for our case



$$\partial_\mu^2 \phi = \phi(1-\phi^2)(4\phi^2-1).$$

The boundary value problem for this case will look differently, since the boundary conditions change, then we get

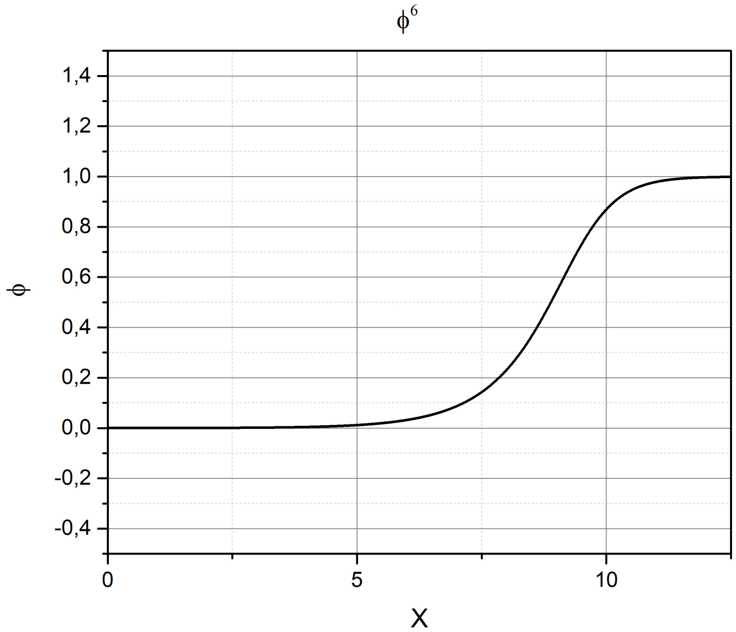
$$\begin{cases} \partial_\mu^2 \phi = \phi(1-\phi^2)(4\phi^2-1), \\ \phi(-\infty) = 0, \\ \phi(\infty) = 1. \end{cases}$$

For the numerical solution of our problem, we obtain the difference scheme of the original boundary value problem

$$\begin{cases} \frac{\phi_{n-1}-2\phi_n+\phi_{n+1}}{h^2} = \phi_n(1-\phi_n^2)(4\phi_n^2-1), \\ \phi_0 = 0, \\ \phi_N = 1, \end{cases}$$

where $n \in (0, N)$ and h is the grid step

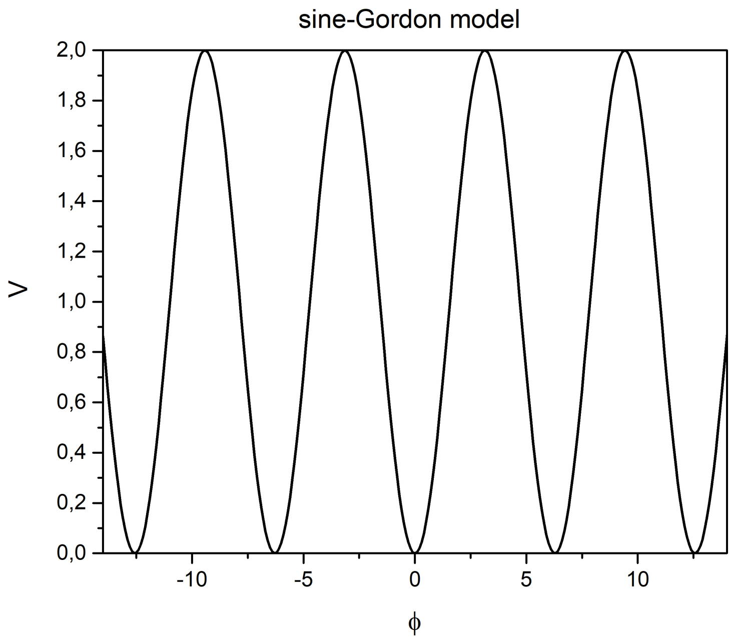
Solving this boundary value problem numerically, we obtain a solution in the form of a kink, which can be represented using the graph:



5 Sine-Gordon model

Another interesting case is models with a periodic potential, one of such models is the sine-Gordon model. The key feature of this model is the existence of an infinite number of vacuum states, but we will look for the desired solution only in one case. First, let's write down the potential of our model :

$$V(\phi) = (1 - \cos\phi).$$



Then the complete Lagrangian of the model can be written in the form

$$L = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - (1 - \cos\phi).$$

Using the Lagrange equation, we obtain the equation of motion for our model :

$$\partial_\mu^2 \phi = -\sin\phi.$$

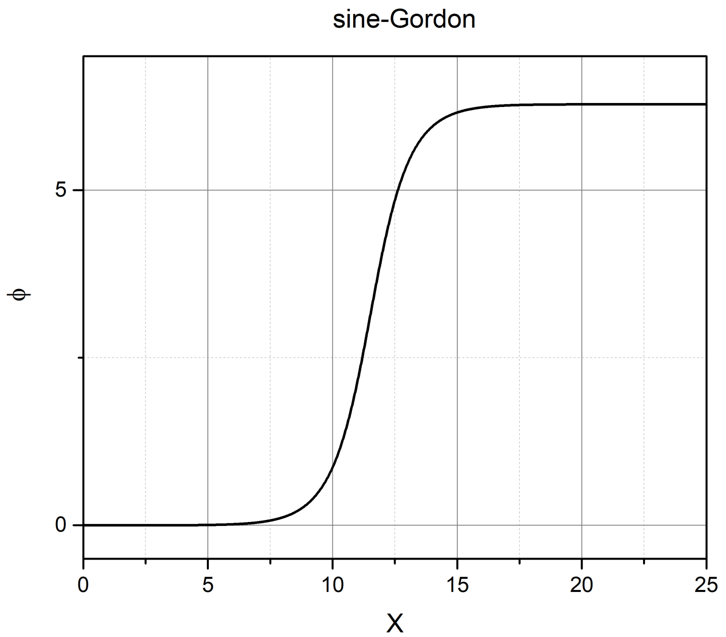
The completely boundary value problem can be formulated as

$$
\begin{cases}
\partial_\mu^2 \phi = -\sin\phi, \\
\phi(-\infty) = 0, \\
\phi(\infty) = 2\pi.
\end{cases}
$$

The difference scheme corresponding to this problem can be written in the form

$$
\begin{cases}
\frac{\phi_{n-1} - 2\phi_n + \phi_{n+1}}{h^2} = -\sin\phi_n, \\
\phi_0(-\infty) = 0, \\
\phi_N(\infty) = 2\pi,
\end{cases}
$$

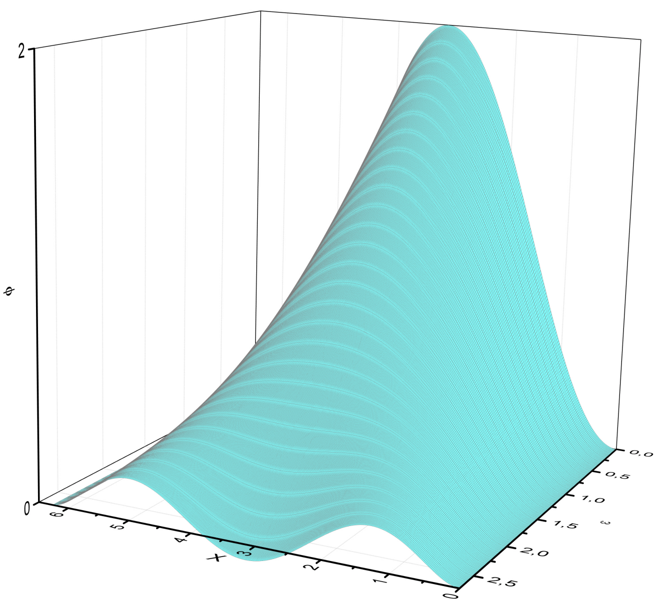
Solving this boundary value problem numerically, we obtain a solution in the form of a kink, which can be represented using the graph



6 Model with parameter

Now let is move on to a more interesting case. In the previous sections, we considered the periodic and polynomial potentials separately. Let us now consider the superposition of these two potentials, but now we have a coefficient ϵ responsible for the contribution of each of the potential:

$$V(\phi) = (1-\epsilon)(1-\cos\phi) + \frac{\epsilon\phi^2}{8\pi^2}(\phi - 2\pi)^2.$$



However, in this case, according to the problem statement, the value of the parameter changes in the range from 0 to 2.7. However, the extreme value at which the model has a solution is 2.6, so from the properties of the system we found out that the model has a physical meaning only for values of the parameter ϵ from 0 to 2.6 . Now we write down the

Lagrangian of our model

$$L = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - (1-\epsilon)(1-\cos\phi) + \frac{\epsilon\phi^2}{8\pi^2}(\phi - 2\pi)^2.$$

Following our algorithm, we obtain the equation of motion for our model

$$\partial_\mu^2 \phi = (\epsilon - 1)\sin\phi + \frac{\epsilon\phi}{2\pi^2}(\phi - 2\pi)(\phi - \pi).$$

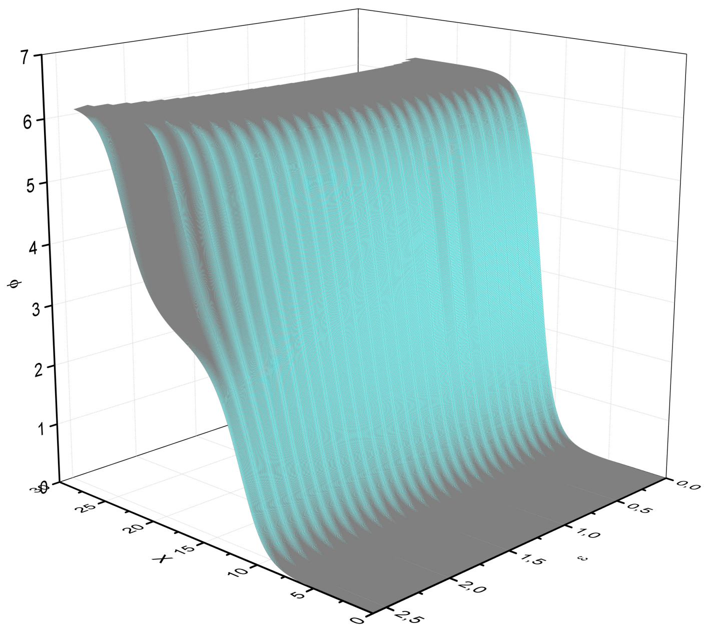
Using the condition for the minimum values and carrying out the appropriate analysis, we find that the system reaches the vacuum state at the values of the argument 0 and 2π. Then, as a result, the boundary value problem can be written in the form

$$\begin{cases} \partial_\mu^2 \phi = (\epsilon - 1)\sin\phi + \frac{\epsilon\phi}{2\pi^2}(\phi - 2\pi)(\phi - \pi), \\ \phi(-\infty) = 0, \\ \phi(\infty) = 2\pi. \end{cases}$$

The complete difference scheme with boundary conditions can be represented as

$$\begin{cases} \frac{\phi_{n-1} - 2\phi_n + \phi_{n+1}}{h^2} = (\epsilon - 1)\sin\phi_n + \frac{\epsilon\phi_n}{2\pi^2}(\phi_n - 2\pi)(\phi_n - \pi), \\ \phi_0(-\infty) = 0, \\ \phi_N(\infty) = 2\pi. \end{cases}$$

Solving this boundary value problem numerically, we obtain a solution in the form of a kink, which can be represented using the graph



7 References

[1] Ya Shnir, ”Topological and Non-Topological Solitons in Scalar Field Theories”, Cambridge University Press, 2018

[2] Kalitkin N. N., "Numerical methods", 1978