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FINAL REPORT ON THE INTEREST PROGRAMME

Numerical Methods in Theory of Topological Solitons

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Abstract

We discuss the concept of topological solitons, and Derrick's theorem. We then study two models; first of which is the O(3) Non-linear Sigma Model in (2+1) dimensions. We discuss its scale invariance, and use it to construct multi-soliton/lump solutions and plot the corresponding energy densities, using north pole and south pole stereo-graphic projections. The same physical results (position and orientation) were successfully produced. The second model we consider is the Planar Baby Skyrme Model in (2+1) dimensions. We discuss its stability owing to the additional skyrme and potential terms in the Lagrangian. Then, we numerically solve a Boundary Value Problem using a shooting method to get and plot the radial profile function for different values of the re-scaled mass parameter. Our results show that the function decreases faster as the mass parameter increases.

Acknowledgements

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Contents

Introduction

1. What are Solitons?

Solitons are non-dissipative, non-trivial, finite energy solutions to non-linear classical field equations. They manifest as localized field configurations with a characteristic size (R), and remain unperturbed in collisions. They appear in different areas of physics: Plasma, Condensed Matter Physics, Biophysics, Non-linear Optics, and more.

Topological solitons arise as a result of non-trivial topologies. They owe their stability to topological constraints that appear as a conserved topological charge (Q) -different from Noether charges-. This charge characterizes the field solution; it sets the number of solitons and the lower bound of energy, as we will see later.

2. Conditions for Soliton Solutions

To get the soliton solutions, we begin by examining the Euler-Lagrange equations of the system.

$$\partial_\mu \frac{\partial \ell}{\partial(\partial^\mu \phi)} = \frac{\partial \ell}{\partial \phi} \qquad (1)$$

In order for the solution to these equations to be physically acceptable, it needs to have a finite energy.

$$E \equiv \int d^d x \mathcal{E}(x,t) \qquad (2)$$

where $\mathcal{E} = T_0^0$ is the energy density.

We also need the vacuum states to be stable over time (static), rather than turning to trivial solutions at later times.

$$\lim_{t\to\infty} \max_{all\to x} \mathcal{E}(x,t) \neq 0 \qquad (3)$$

3. Derrick's Theorem

Derrick's theorem states that there are no soliton solutions i.e. no static, finite energy solutions for spatial dimensions greater than 2 for the general Lagrangian density

$$\ell = \frac{1}{2}\partial_\mu \vec{\phi} \partial^\mu \vec{\phi} - U(|\phi|) \qquad (4)$$

Given this Lagrangian, the energy function would be

$$E = \int d^d x [\frac{1}{2}(\partial_k \vec{\phi})^2 + U(|\phi|)] = E_2 + E_0 \qquad (5)$$

Where d is the number of spatial dimensions, $E_2 = \frac{1}{2}\int d^d x (\partial_k \vec{\phi})^2$ and $E_0 = \int d^d x U(|\phi|)$.

Since solitons are localized field configurations with a characteristic size R, scaling deformations $x \to \lambda x$ (λ a positive constant, is the spatial dilation factor) should not affect the global minimum of the energy function, and $R \to \lambda^{-1} R$.

Under this transformation, we find that the energy function transforms as follows

$$E = E_2 + E_0 \to \lambda^{2-d} E_2 + \lambda^{-d} E_0 \qquad (6)$$

If we consider a one-parametric family of field configurations $\vec{\phi}_\lambda(x) = \vec{\phi}(\lambda x)$, and $E(\lambda) = E[\vec{\phi}_\lambda]$. To find the stationary point of the energy function for $\vec{\phi}(x)$, we differentiate the energy function with respect to λ at $\lambda = 1$ and equate to zero.

$$\frac{dE(\lambda)}{d\lambda} = (2-d)\lambda^{1-d} E_2 - d\lambda^{-d-1} E_0 = 0$$
$$\left.\frac{dE(\lambda)}{d\lambda}\right|_{\lambda=1} = (2-d)E_2 - dE_0 = 0 \qquad (7)$$

Therefore, the existence of a stationary point at $\lambda = 1$ depends on d.

For $d = 1$, $E_2 = E_0$. For $d = 2$, $E_0 = 0$ everywhere and E_2 takes any value, $\vec{\phi}(x)$ is always in the vacuum state, and the model becomes scale invariant i.e the size of the soliton can increase or decrease indefinitely with scale deformations, but we can still get static, finite energy solutions (lumps) by choosing the appropriate vacuum boundary conditions, as we will see in the O(3) non-linear sigma model.

Furthermore, by examining the second derivative of the energy function with respect to λ, we get in $d = 1$,

$$\left.\frac{d^2E(\lambda)}{d\lambda^2}\right|_{\lambda=1} = 2E_0 > 0 \qquad (8)$$

meaning that $\lambda = 1$ corresponds to a minimum of the energy function, and the solution is stable with respect to a scaling transformation.

For $d = 2$,

$$\left.\frac{d^2E(\lambda)}{d\lambda^2}\right|_{\lambda=1} = 6E_0 = 0 \qquad (9)$$

meaning that there is no preferred scale, and there will always be a zero mode in the radial fluctuations spectrum causing instability (scale invariance).

For $d > 2$, we will have

$$\left.\frac{d^2E(\lambda)}{d\lambda^2}\right|_{\lambda=1} < 0 \qquad (10)$$

meaning that the energy is unstable, and decreases as the solitons shrink. Thus proving Derrick's theorem.

There are multiple ways to evade Derrick's theorem and to stabilize the solution. We will discuss one of those ways in the baby skyrme model.

4. O(3) Non-Linear Sigma Model

The O(3) non-linear sigma model is a scale invariant model; our solutions are not stable as the size of the soliton can increase or decrease indefinitely under scale deformations. Therefore, it is not very accurate to call them solitons, and so the proper nomenclature that we will be using henceforth is “lumps”.

If we consider real scalar fields in (d+1)-dimensional Minkowski space, constrained to the unit sphere, then the constrained Lagrangian will be

$$\ell = \frac{1}{4}\partial_\mu \phi^a . \partial^\mu \phi^a + \lambda(1 - \phi^a . \phi^a) \qquad \textit{(11)}$$

where $a = 1,2,n$ and ϕ^a are real scalar fields with the constraint $\phi^a . \phi^a = 1$ enforced by the Lagrange multiplier λ.

The resulting Euler-Lagrange equations after eliminating λ will be

$$\partial_\mu \partial^\mu \phi^a + (\partial_\mu \phi^b . \partial^\mu \phi^b)\phi^a = 0 \qquad \textit{(12)}$$

If we restrict our treatment to (2+1)-dimensions and a triplet of real scalar fields, we get the energy function

$$E = \int d^2x(\partial_i \phi^a . \partial_i \phi^a) \qquad \textit{(13)}$$

where $i = 1,2$, $a = 1,2,3$. For the energy to be finite, ϕ must tend to a constant vector at infinity if our space is a plane. Equivilantly, for our spherical space, we can take this vector as (0,0,1) i.e the north pole of the sphere, which represents our infinity if we map the sphere onto a plane using a north pole stereo-graphic projection.

This boundary condition breaks the O(3) symmetry to an O(2) symmetry (ϕ_1 and ϕ_2 rotations), which yields the compactification of the domain space from $\boldsymbol{R}^2$ to S^2, and the field becomes a map $\phi : S^2 \to S^2$ from physical to target space, where the relevant homotopy group $\pi_2(S^2) = Z$. This implies that the topological configuration is characterized by an integer called the topological degree/charge Q.

This topological charge Q as discussed before can be considered as the number of lumps in the configuration, and is given by

$$Q = \frac{1}{8\pi}\int d^2x \varepsilon_{abc} \varepsilon_{ij} \phi^a \partial_i \phi^b \partial_j \phi^c \qquad \textit{(14)}$$

It also sets the lower bound of energy (Bogomolny bound)

$$E \geq \pm 4\pi Q \qquad \textit{(15)}$$

which is saturated if and only if

$$\partial_i \phi^a \partial_i \phi^a = \varepsilon_{abc} \varepsilon_{ij} \phi^a \partial_i \phi^b \partial_j \phi^c \quad \textit{(16)}$$

These are called Bogomolny equations.

It is more convenient to make the following changes of variables

$$W = \frac{\phi_1 + i\phi_2}{1+\phi_3}, \qquad (17)$$
$$z = x + iy,$$

where W is is a map from the complex plane to the Riemannian sphere of (ϕ_1, ϕ_2, ϕ_3) and can be taken as a function of z and $\bar{z}$.

Consequently, the Lagrangian density, energy function and topological charge become

$$\ell = \frac{\partial_\mu W \partial^\mu \overline{W}}{(1+|W|^2)^2}$$
$$E = \int \frac{|W_z|^2 + |W_{\bar{z}}|^2}{(1+|W|^2)^2} dz d\bar{z} \qquad (18)$$
$$Q = \frac{1}{4\pi} \int \frac{|W_z|^2 - |W_{\bar{z}}|^2}{(1+|W|^2)^2} dz d\bar{z}$$

where $W_z = \frac{\partial W}{\partial z}$ and $W_{\bar{z}} = \frac{\partial W}{\partial \bar{z}}$. This is known as the ***CP₁*** model.

The minimum of the energy function correspond to the following cases (Bogomolny approach)

$$W_{\bar{z}} = 0$$
$$Q = 4\pi E \qquad (19)$$

which means that W is a holomorphic function, and

$$W_z = 0$$
$$Q = -4\pi E \qquad (20)$$

which means that W is an anti-holomorphic function.

Taking W to be holomorphic i.e Q is positive, and since W is required to have a definite value as $z \to \infty$, and the total energy is finite, therefore W is a rational map i.e

$$W(z) = \frac{p(z)}{q(z)} \qquad (21)$$

where the topological charge Q equals the highest power of z in either p or q.

This simple result allows us to construct multi-soliton/lump configurations easily by choosing any form for p and q ! It also allows us to control their positions and orientations!

5. Baby/Planar Skyrme Model

One of the ways we could stabilize the O(3) model is to add extra terms to the Lagrangian that scale both as positive and negative powers of the spatial dilation factor λ. Consider the following Lagrangian for the baby skyrme model:

$$\ell = \frac{1}{2}(\partial_\mu \phi^a)^2 - \frac{1}{4}(\varepsilon_{abc}\phi^a \partial_\mu \phi^b \partial_\nu \phi^c)^2 - U(|\phi|) \qquad \textit{(22)}$$

The first term is the normal O(3) model term, the second term is called the skyrme term, and the final term is a potential term.

The energy function will be

$$E = \int d^2x[\frac{1}{2}(\partial_i \phi^a)^2 + \frac{1}{4}(\varepsilon_{abc}\phi^a \partial_i \phi^b \partial_j \phi^c)^2 + U(|\phi|)] \qquad \textit{(23)}$$

and under the scaling transformation

$$E = E_4 + E_2 + E_0 \to \lambda^{-2} E_4 + E_2 + \lambda^2 E_0 \qquad \textit{(24)}$$

The potential term scales inversely to the skyrme term, thus stabilizing the solutions that satisfy the relation $E_4 = E_0$.

One of the choices for the potential term is to take $U = \mu^2(1-\phi_3)$ where μ is the re-scaled mass parameter. This potential breaks the O(3) symmetry to O(2) as there is a single vacuum state at $\phi_3 = 1$. The final term now gives the mass of the fields ϕ_1 and ϕ_2, and ϕ_3 remains mass-less.

To be finite, the energy function must approach the vacuum state (0,0,1) as r goes to infinity, and the energy will still have a lower bound as equation *(15)* before $E \geq \pm 4\pi Q$.

However, this time, the lower bound will never be saturated, and thus we must solve the field equations directly.

We make use of the Lagrange multiplier to enforce the constraint $\phi^a.\phi^a = 1$. After eliminating it as before, we arrive to the field equation that we are required to solve, which is a messy non-linear, second order, partial differential equation.

To simplify it, we make use of the O(2) symmetry of the model, and consider the hedgehog ansatz for $Q = 1$:

$$\begin{aligned}\phi_1 &= \cos(\theta)\sin(f(r)) \\ \phi_2 &= \sin(\theta)\sin(f(r)) \qquad \textit{(25)} \\ \phi_3 &= \cos(f(r))\end{aligned}$$

where $f(r)$ is a monotonically decreasing radial profile function. Since the field must approach the vacuum state at infinity then $\cos(f(r)) \to 1$ as $r \to \infty$ or $f(\infty) \to 0$.

Substituting equations *(24)* into equation *(23)*, we get

$$E = 2\pi \int r dr [\frac{1}{2} f'^2 + \frac{\sin^2(f)}{2r^2}(f'^2 + 1) + \mu^2 (1 - \cos(f))] \qquad (26)$$

and therefore we get the following variational equation that corresponds to the stationary point of the energy function

$$(r + \frac{\sin^2(f)}{r}) f'' + (1 - \frac{\sin^2(f)}{r^2} + \frac{f' \sin(f)\cos(f)}{r}) f' - \frac{\sin(f)\cos(f)}{r} - r\mu^2 \sin(f) = 0 \qquad (27)$$

The topological charge from equation *(14)* becomes (for $Q = 1$)

$$Q = \frac{1}{2}\int_0^\infty r dr (\frac{f' \sin(f)}{r}) = \frac{1}{2}[\cos(f(\infty)) - \cos(f(0))] = 1 \qquad (28)$$

This gives us the boundary conditions

$$\begin{aligned} f(0) &= \pi \\ f(\infty) &= 0 \end{aligned} \qquad (29)$$

Equations *(27)* and *(28)* make a boundary value problem that can be solved numerically to find a solution for the profile function $f(r)$.

Project Objectives

During this project, we had two objectives in mind; One for each model discussed above. Those objectives were:

1. **For the O(3) sigma model:**
   Our goal was to construct multi-soliton/lump solutions from equations *(21)* and *(18)*, using a south pole stereo-graphic projection, having the same position and orientation as ones constructed using a north pole stereo-graphic projection.
2. **For the baby skyrme model:**
   Our goal was to find a numerical solution for the profile function from equations *(28)* and *(29)* as a Boundary Value Problem BVP, and then plot it for different values of the re-scaled mass parameter μ.

Scope of Work and Methods

- **O(3) Sigma Model**

To construct soliton solutions, we make use of the following equations:

(1) $W(z)=\dfrac{p(z)}{q(z)}$, this rational map represents a specific field configuration i.e soliton solution (ϕ_1,ϕ_2,ϕ_3).

(2) $\mathcal{E}=\dfrac{|W_z|^2+|W_{\bar{z}}|^2}{(1+|W|^2)^2}$, this is the energy density of the configuration.

We can choose any form for p and q, where the highest power of z in either will represent the number of lumps in the configuration.

To construct solutions with the same position and orientation using north pole and south pole stereo-graphic projections, we make the following table:

North Pole Stereographic Projection	South Pole Stereographic Projection
$W_n=\dfrac{\phi_1+i\phi_2}{1-\phi_3}$	$W_s=\dfrac{\phi_1-i\phi_2}{1+\phi_3}$
$W_n=\dfrac{1}{W_s}$	$W_s=\dfrac{1}{W_n}$
$\mathcal{E}_n=\dfrac{\lvert W_{n_z}\rvert^2}{(1+\lvert W_n\rvert^2)^2}$	$\mathcal{E}_s=\dfrac{\lvert W_{s_z}\rvert^2}{(1+\lvert W_s\rvert^2)^2}$

*Note the orientation reversal by complex conjugation of W to maintain consistent physical orientation of the solution.

*Note that the transition map between the two projections is $f(z)=\dfrac{1}{z}$.

*Note that W is holomorphic for $Q=+ve$, so $W_{\bar{z}}=0$.

Method:

We used Mathematica to plot the energy density for different W(z) for both projections:

(1) For $Q=1$, One-soliton/lump solution, $W(z)=\dfrac{(z-1)}{(z-2)}$.

(2) For $Q=2$, Two-solitons/lumos solution, $W(z)=\dfrac{(z-4)(z+4)}{(z-1)(z+1)}$.

(3) For $Q=8$, Eight-soliton/lumps solution,

$$W(z)=\frac{4}{\frac{1}{z}+\frac{1}{z+0.5-i}+\frac{1}{z-0.5-i}+\frac{1}{z-1}+\frac{1}{z+1}+\frac{1}{z+1.5+i}+\frac{1}{z-1.5+i}+\frac{1}{z-2i}}$$

- **Baby Skyrme Model**

To find a numerical solution for the profile function, we use the following equations

(1) $$(r+\frac{\sin^2(f)}{r})f''+(1-\frac{\sin^2(f)}{r^2}+\frac{f'\sin(f)\cos(f)}{r})f'-\frac{\sin(f)\cos(f)}{r}-r\mu^2\sin(f)=0$$

(2) $$f(0)=\pi$$
$$f(\infty)=0$$

to construct a boundary value problem.

Method:

We used the NDSolve function in Mathematica, using a shooting method to solve the BVP using $\mu = 0.1$, and then we plotted the results for different values of μ.

*Note: We took the second boundary value to be $f(8)=0$ and $f(30)=0$ to be able to do the computation and to compare the plots of the two values.

We show the results for both models in the next section.

Plots and Results

- **O(3) Sigma Model**

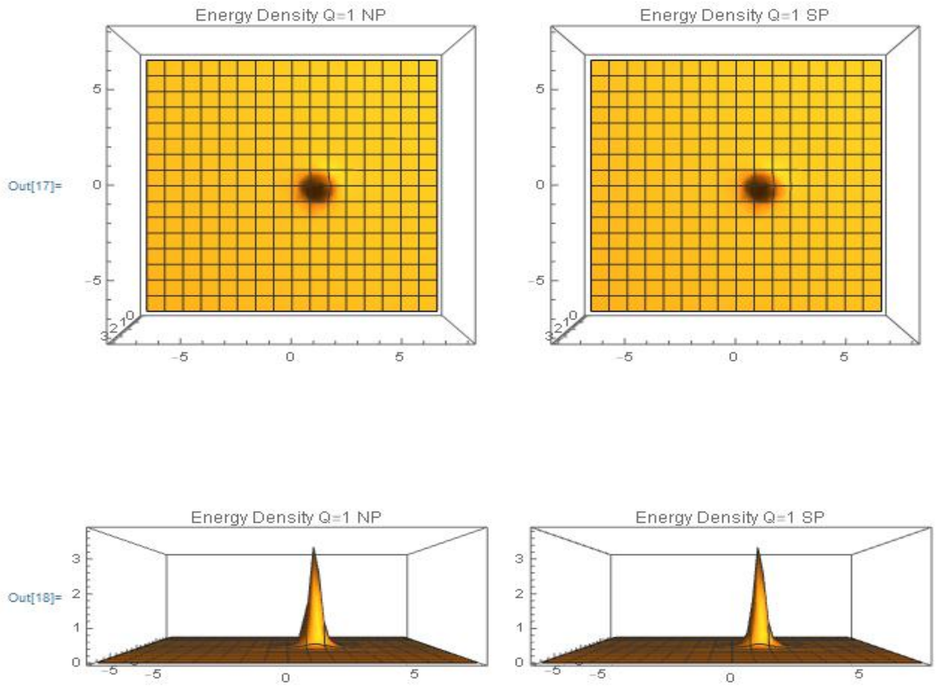

Figure (1). Plots for $Q=1$ viewed from above (first row) and from the front (second row), using North Pole projections (first column) and south pole projections (second column)

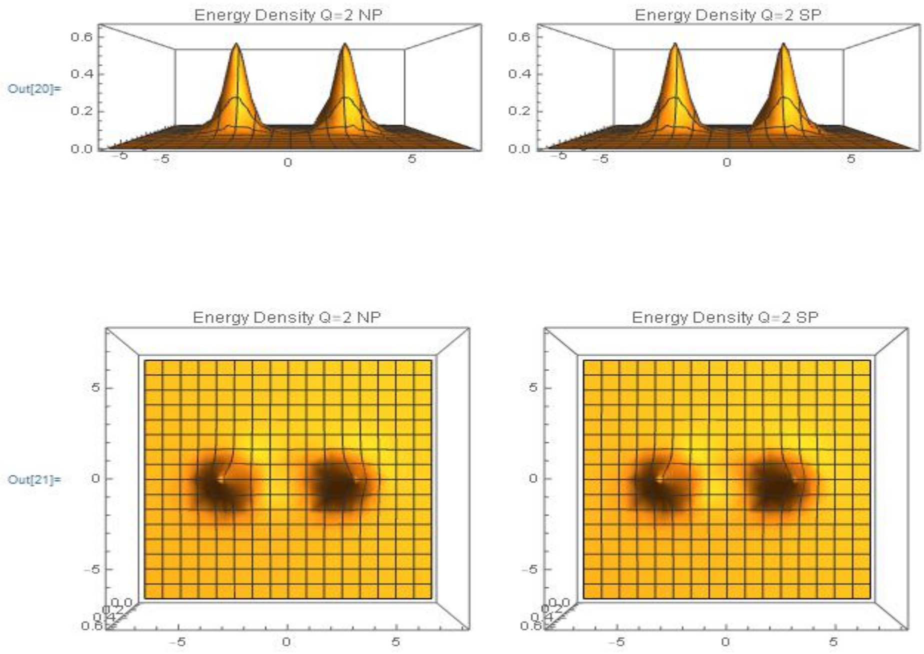

Figure (2). Plots for $Q=2$ viewed from the front (first row) and from above (second row), using North Pole projections (first column) and south pole projections (second column)

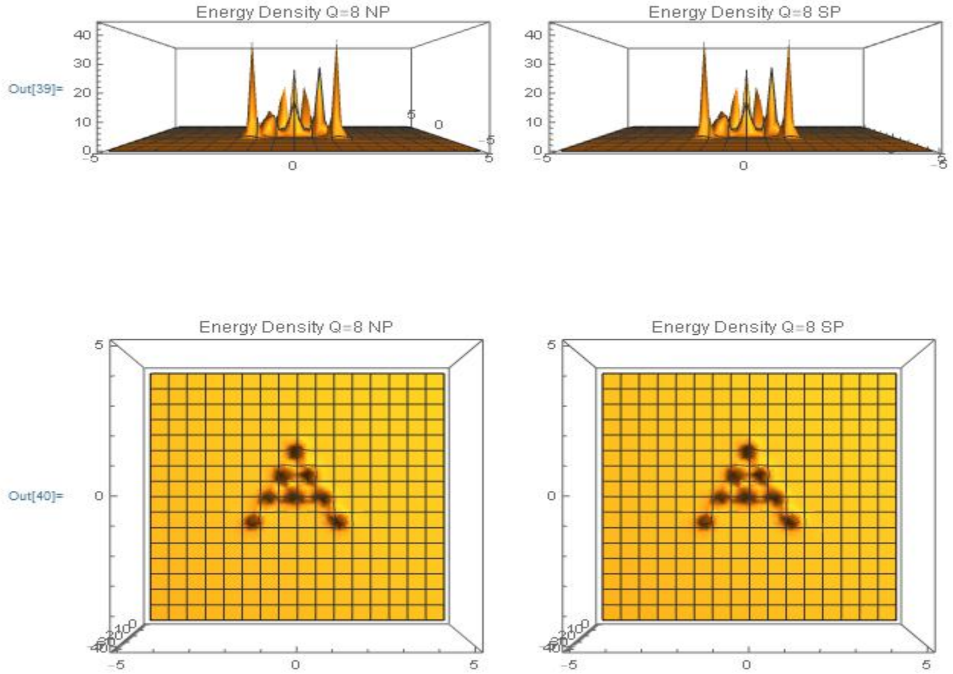

Figure (3). Plots for $Q=8$ viewed from the front (first row) and from above (second row), using North Pole projections (first column) and south pole projections (second column)

- **Baby Skyrme Model**

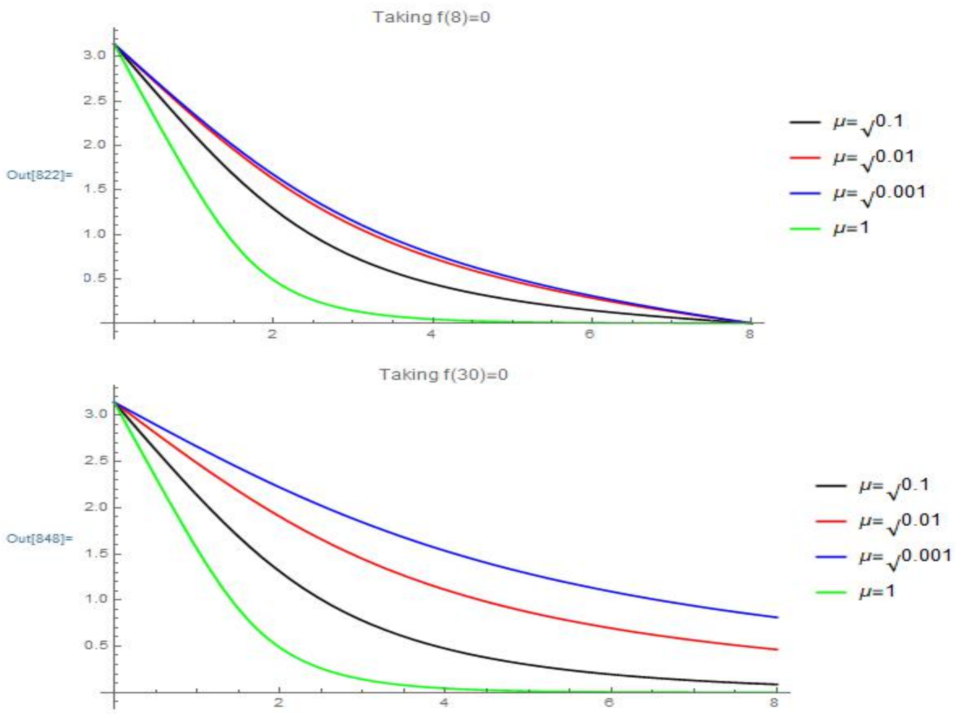

Figure (4). Plots for $f(r)$ for different values of re-scaled mass μ, taking $f(8)=0$ (above) and $f(30)=0$ (below)

Conclusion

At the end of the project, we managed to successfully produce the desired plots. For the O(3) sigma model, we managed to produce the same physical results (position and orientation) using a north pole stereo-graphic projection and a south pole stereo-graphic projection. We could now go on to compare the charge density plots as well using the same procedure.

For the baby skyrme model, we managed to compare the numerical solutions for different re-scaled mass parameter μ values. We found that the radial profile function decreases faster as μ increases. We could go on to compare how different numerical methods of solving the equation behave, and compare accuracies and errors between them. We could also go on to construct different baby skyrmion solutions, or use different forms of the potential term. The options are endless!

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